9 Elliptic PDE's

We will apply the results of singular integrals particularly the estimate that the Riesz transforms are bounded on evry $L_p(R^d)$ for $1 < p < \infty$ to prove existence of solutions $u \in W_{2,p}(R^d)$ for the equation

$$u(x) - \sum_{i,j} a_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_j b_j(x) \frac{\partial u}{\partial x_j} = f(x)$$

provided $f \in L_p$ and the coefficients of

$$L = \sum_{i,j} a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_j b_j(x) \frac{\partial}{\partial x_j}$$

satisfy

1. The coefficients $\{a_{i,j}(x)\}$, (assumed to satisfy with out loss of generality the symmetry condition $a_{i,j}(x) \equiv a_{j,i}(x)$), are unfiformly continuous on R^d and satisfy

$$c \sum_j \xi_j^2 \leq \sum_{i,j} a_{i,j}(x) \xi_i \xi_j \leq C \sum_x i_j^2 \tag{9.1}$$

for some $0 < c \leq C < \infty$.

2. The coefiicients $\{b_j(x)\}$ are measurable and satisfy

$$\sum_j |b_j(x)|^2 \leq C < \infty \tag{9.2}$$

We first derive apriori bounds. We asume that p is arbitrary in the range $1 < p < \infty$ but fixed. Let A_p be a bound for the Riesz transforms in $L_p(R^d)$. If we look at all constant coefficent operators

$$L_Q = \sum q_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}$$

with symmetric matrces Q satsfying the bounds (9.1) by a linear transformation they can be reduced to the operator Δ and if $\Delta u = f$ and $f \in L_p(R^d)$ we have the bounds

$$\|u_{x_i,x_j}\|_p \leq A_p^2 \|f\|_p$$

and factoring in the constants coming from the linear transformation we can still conclude that there is a constant $A = A(p, c, C, d)$ such that if $L_Q u = f$, then

$$\|u_{x_i,x_j}\|_p \leq A\|f\|_p$$

Lemma 9.1. *If $\epsilon \leq \epsilon_0$ is small enough and $\sup_{x \in R^d} |a_{i,j}(x) - q_{i,j}| \leq \epsilon$ for some Q satisfying (9.1) we can still conclude that for any $u \in W_{2,p}$ that satisfies*

$$\sum a_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) = f(x)$$

we must necessarily have a bound

$$\|u_{x_i,x_j}\|_p \leq C\|f\|_p$$

for some $C = C(A, d, \epsilon_0)$ independently of u. Consequently if u is supported in a ball where $|a_{i,j}(x) - q_{i,j}| \leq \epsilon_0$ and

$$\sum a_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) = f(x)$$

then again

$$\|u_{x_i,x_j}\|_p \leq C\|f\|_p$$

Proof. Let us compute

$$\begin{aligned} L_Q u = \sum_{i,j} q_{i,j} u_{i,j} &= \sum_{i.j} a_{i,j}(x) u_{i,j}(x) - \sum_{i,j} [a_{i,j}(x) - q_{i.j}] u_{i,j}(x) \\ &= f - \sum_{i,j} \epsilon_{i,j}(x) u_{i,j}(x) \end{aligned}$$

$$\|L_Q u\|_p \leq \|f\|_p + \epsilon_0 d^2 \sup_{i,j} \|u_{i,j}\|_p$$

On the other hand

$$\sup_{i,j} \|u_{i,j}\|_p \leq A\|L_Q u\|_p \leq A\|f\|_p + A\epsilon_0 d^2 \sup_{i,j} \|u_{i,j}\|_p$$

If ϵ_0 is chosen so that $A\epsilon_0 d^2 \leq \frac{1}{2}$, then

$$\sup_{i,j} \|u_{i,j}\|_p \leq 2A\|f\|_p$$

For the second part we alter the coeffecients outside the support of u so that we are back in a situation where we can apply the first part. □

We now consider a ball of radius $\delta < 1$ small enough that if x_0 is the center of the ball and x is any point in the ball, then $|a_{i,j}(x) - a_{i,j}(x_0)| \le \epsilon_0$. This is possible because of uniform continuity of the coeffecients $\{a_{i,j}(x)\}$. Let B_δ be such a ball, and let

$$Lu = f \text{ in } B_\delta$$

Theorem 9.1. *There is a constant C such that for any $\rho < 1$*

$$\sup_{i,j} \|u_{i,j}\|_{p,B_{\rho\delta}} \le C[\|f\|_{p,B_\delta} + \delta^{-1}(1-\rho)^{-1}\|\nabla u\|_{p,B_\delta} + \delta^{-2}(1-\rho)^{-2}\|u\|_{p,B_\delta}] \tag{9.3}$$

Proof. Let us for the moment take $\delta = 1$ and construct a smooth function $\phi = \phi_\rho$ such that $\phi = 1$ on B_ρ and 0 outside B_1. We can assume that $|\nabla\phi| \le C(1-\rho)^{-1}$ and $|\nabla\nabla\phi| \le C(1-\rho)^{-2}$. We take $v = u\phi$ and compute

$$\begin{aligned} g &= \sum_{i,j} a_{i,j}(x)v_{i,j}(x) = \sum_{i,j} a_{i,j}(x)(\phi u)_{i,j}(x) \\ &= \phi \sum_{i,j} a_{i,j}(x)u_{i,j}(x) + 2\sum a_{i,j}(x)\phi_i(x)u_j(x) + u(x)\sum_{i,j} a_{i,j}(x)\phi_{i,j}(x) \\ &= \phi(x)f(x) - \phi(x)\sum b_j(x)u_j(x) + 2\sum a_{i,j}(x)\phi_i(x)u_j(x) \\ &\qquad + u(x)\sum_{i,j} a_{i,j}(x)\phi_{i,j}(x) \end{aligned}$$

We can bound

$$|g| \le |f(x)| + C(1-\rho)^{-1}\|\nabla u\|(x) + C(1-\rho)^{-2}|u|(x)$$

From the previous lemma we can get

$$\sup_{i,j} \|v_{i,j}\|_{p,B_1} \le A\|g\|_{p,B_1} \le C[\|f\|_{p,B_1} + (1-\rho)^{-1}\|\nabla u\|_{p,B_1} + (1-\rho)^{-2}\|u\|_{p,B_1}]$$

Since $v = u$ on B_ρ we get

$$\sup_{i,j} \|u_{i,j}\|_{p,B_\rho} \le C[\|f\|_{p,B_1} + (1-\rho)^{-1}\|\nabla u\|_{p,B_1} + (1-\rho)^{-2}\|u\|_{p,B_1}]$$

If $\delta < 1$ we can redefine all functions involved as $u(\delta x), f(\delta x), a_{i,j}(\delta x)$ and $\delta b_j(\delta x)$. With the new operator

$$L_\delta = \sum a_{i,j}(\delta x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum \delta b_j(\delta x)\frac{\partial}{\partial x_j}$$

we see that

$$L_\delta u(\delta x) = \delta^2 f(\delta x)$$

We can now apply our estimate with $\delta = 1$ and obtain

$$\sup_{i,j} \|u_{i,j}\|_{p,B_{\delta\rho}} \le C[\|f\|_{p,B_\delta} + \delta^{-1}(1-\rho)^{-1}\|\nabla u\|_{p,B_\delta} + \delta^{-2}(1-\rho)^{-2}\|u\|_{p,B_\delta}]$$

□

At this point we can do one of two things. If we are interested only in dealing with all of R^d we can raise the estimate (9.3) to the power p and sum over a fine enough grid so that

$$0 < a < \sum_\alpha \mathbf{1}_{B(x_\alpha,\delta\rho)} \le \sum_\alpha \mathbf{1}_{B(x_\alpha,\delta)} \le A < \infty$$

and we will get

$$\sup_{i,j} \|u_{i,j}\|_p \le C[\|f\|_p + \delta^{-1}(1-\rho)^{-1}\|\nabla u\|_p + \delta^{-2}(1-\rho)^{-2}\|u\|_p]$$

Since $\delta > 0$ is fixed (depending on the modulus of continuity of $\{a_{i,j}(x)\}$) and we could have fixed $\rho = \frac{1}{2}$, we have the following global estimate for any $u \in W_{2,p}$ satisfyng $Lu = f$. The constant C depends only on the ellipticity bounds in (9.1), the bounds in (9.2) and the modulus of continuty estimates of $\{a_{i,j}(x)\}$.

$$\sup_{i,j} \|u_{i,j}\|_p \le C[\|f\|_p + \|\nabla u\|_p + \|u\|_p] \tag{9.4}$$

Lemma 9.2. *For any constant $\epsilon > 0$, there is a constant C_ϵ such that for any $u \in W_{2,p}$*

$$\|\nabla u\|_p \le \epsilon \sup_{i,j} \|u_{i,j}\|_p + C\epsilon^{-1}\|u\|_p \tag{9.5}$$

Proof. First we note that it is sufficient to prove an estimate of the type

$$\|\nabla u\|_p \le C[\sup_{i,j} \|u_{i,j}\|_p + \|u\|_p] \tag{9.6}$$

We can then replace $u(x)$ by $u(\lambda x)$ and the estimate takes the form

$$\lambda\|\nabla u\|_p \le C[\lambda^2 \sup_{i,j} \|u_{i,j}\|_p + \|u\|_p]$$

If choosing $\lambda = C\epsilon$ the lemma is seen to be true. To prove (9.6) we basically need a one dimensional estimate. If we have

$$\int_{-\infty}^{\infty} |g'(x)|^p dx \leq C \int_{-\infty}^{\infty} |g''(x)|^p dx + C \int_{-\infty}^{\infty} |g(x)|^p dx$$

on R, we could get the estimate on each line and then integrate it. The inequality itself needs to be proved only for the unit interval $[0,1]$. We can then translate and sum. It is quite easy to prove

$$\sup_{0\leq x\leq 1} |g'(x)| \leq C[\int_0^1 |g''(x)|dx + \int_0^1 |g(x)|dx]$$

□

Our basic apriori estmate becomes

Theorem 9.2. *Any function $u \in W_{2,p}$ with $Lu = f$ satisfies*

$$\sup_{i,j} \|u_{i,j}\|_p \leq C[\|f\|_p + \|u\|_p] \tag{9.7}$$

Proof. Just choose ϵ in (9.5) so that $C\epsilon < \frac{1}{2}$ where C is the constant in (9.4). □

We have to work a little harder If we want to prove a local regularity estimate of the form

Theorem 9.3. *Let $\Omega \subset \bar{\Omega} \subset \Omega'$ be bounded sets. For any $u \in W_{2,p}(\Omega')$ with $Lu = f$, we have the bounds*

$$\|u_{i,j}\|_{p,\Omega} \leq C(\Omega, \Omega')[\|f\|_{p,\Omega'} + \|u\|_{p,\Omega'}] \tag{9.8}$$

Proof. The trick is to go back and change the definition of ϕ_ρ so that it vanishes outside the ball of radius $\frac{1+\rho}{2}$ rather than outside the ball of radius 1. It does not change much since $(1 - \frac{1+\rho}{2}) = \frac{1}{2}(1-\rho)$. We start with the modified version of (9.3)

$$\sup_{i,j} \|u_{i,j}\|_{p,B_\rho} \leq C[\|f\|_{p,B_1} + (1-\rho)^{-1}\|\nabla u\|_{p,B_{1+\frac{\rho}{2}}} + (1-\rho)^{-2}\|u\|_{p,B_1}]$$

and define

$$T_2 = \sup_{\frac{1}{2}<\rho<1} (1-\rho)^2 \sup_{i,j} \|u_{i,j}\|_{p,B_\rho}$$
$$T_1 = \sup_{\frac{1}{2}<\rho<1} (1-\rho)\|\nabla u\|_{p,B_\rho}$$
$$T_0 = \sup_{\frac{1}{2}<\rho<1} \|u\|_{p,B_\rho} = \|u\|_{p,B_1}$$

We see that

$$T_2 \le C[\|f\|_{p,B_1} + T_1 + T_0]$$

Assume a uniform interpolation inequality for all balls of radius $\frac{1}{2} \le \rho \le 1$ of the type,

$$\|\nabla u\|_{p,B_\rho} \le \epsilon \sup_{i,j} \|u_{i,j}\|_{p,B_\rho} + C\epsilon^{-1}\|u\|_{p,B_\rho}$$

for any choice of $\epsilon > 0$, that translates to

$$T_1 \le \epsilon T_2 + C\epsilon^{-1} T_0$$

and with the right chice of ϵ we get

$$T_2 \le C[\|f\|_{p,B_1} + \|u\|_{p,B_1}]$$

In particular

$$\|u_{i,j}\|_{p,B_\rho} \le C(1-\rho)^{-2}\ [\|f\|_{p,B_1} + \|u\|_{p,B_1}]$$

With rescaling for $\delta_1 < \delta_2 < \delta_0$,

$$\|u_{i,j}\|_{p,B_{\delta_1}} \le C(\delta_1,\delta_2)\ [\|f\|_{p,B_{\delta_2}} + \|u\|_{p,B_{\delta_2}}]$$

Covering $\bar{\Omega}$ by a finite number of balls of radius δ_1, such that the concentric balls of radius δ_2 are still contained in Ω' we get our result. □

Finally we prove the interpolation lemma for balls.

Lemma 9.3. *Given $u \in W_{2,p,B_1}$ it can be extended as a function v on R^d supported on B_2 such that*

$$\|\nabla\nabla v\|_{p,R^d} \le C[\|\nabla\nabla u\|_{p,B_1} + \|\nabla u\|_{p,B_1}]$$
$$\|\nabla v\|_{p,R^d} \le C\|\nabla u\|_{p,B_1}$$
$$\|v\|_{p,R^d} \le C\|u\|_{p,B_1}$$

Proof. Basically if we want a function which is smooth inside B_1 and outside B_1 to be globally in $W_{2,p}$ the function and its derivatives have to match on the boundary. The usual reflection with $v(1+r,s) = u(1-r,s)$ for small r matches the function and tangential derivatives but not the normal derivative. $v(1+r,s) = c_1 u(1-r,s) + c_2 u(1-2r,s)$ works for a proper choice of c_1 and c_2. We use it to extend to $B_{\frac{3}{2}}$ and then a radial cutoff to kill it outside B_2. For the extended function v we have the interpolation inequality

$$\|\nabla v\|_{p,R^d} \leq \epsilon \|\nabla\nabla u\|_{p,R^d} + C\epsilon^{-1}\|u\|_{p,R^d}$$

and this implie for the original u

$$\|\nabla u\|_{p,B_1} \leq C\epsilon \|\nabla\nabla u\|_{p,B_1} + C\epsilon\|\nabla u\|_{p,B_1} + C\epsilon^{-1}\|u\|_{p,B_1}$$

which is easily turned into

$$\|\nabla u\|_{p,B_1} \leq \epsilon \|\nabla\nabla u\|_{p,B_1} + C\epsilon^{-1}\|u\|_{p,B_1}$$

□

Finally we prove an existence theorem for solutions of $u - Lu = f$.

Theorem 9.4. *The equation*

$$u - Lu = f$$

has a solution in $W_{2,p}$ for each $f \in L_p$.

Proof. We wish to invert $(I - L)$. Suppose we can invert $(I - L_1)$. Then

$$(I-L_2)^{-1} = [(I-L_1)-(L_2-L_1)]^{-1} = [I-L_1]^{-1}[I-(L_2-L_1)(I-L_1)^{-1}]^{-1}$$

As long as $\|(L_2 - L_1)(I - L_1)^{-1}\| < 1$ as an operator mapping $L_p \to L_p$, $(I - L_2)^{-1}$ will map L_p into $W_{2,p}$. We can perturb the operators from Δ to any L nicely in small steps so that $\|L_1 - L_2\| < \delta$ as operators from $W_{2,p} \to L_p$. All we need are uniform apriori bounds on $\|(I - L)^{-1} f\|_p$. □

Theorem 9.5. *Any solution u of $u - Lu = f$ with L satisfying (9.1) and (9.2) also satisfies a bound of the form*

$$\|u\|_p \leq C\|f\|_p$$

with a constant that does not depend on L or f.

The proof depend on lemmas.

Lemma 9.4 (Maximum Principle). *Suppose $u \in W_{2,p}$ satisfies $u - Lu \geq 0$ in a possibly unbounded region G and p is large enough that Sobolev imbedding applies and u is bounded and continuous on $\bar{G}$. If in addition $u \geq 0$ on ∂G then $u \geq 0$ on $\bar{G}$. In particular if u and v are two functions wth $u - Lu \geq v - Lv$ in G and $u \geq v$ on ∂G, then $u \geq v$ on $\bar{G}$.*

Lemma 9.5. *If $u - Lu = 0$ in a ball $B(x, \delta)$ of radius δ, then*

$$|u(x)| \leq \rho(\delta) \sup_{y:|y-x|=\delta} |u(y)|$$

Proof. We can cssume with out loss of generality that $x = 0$. Consider the function

$$\phi(x) = \exp[-c(1 - \frac{|x|^2}{\delta^2})]$$

For some $c = c(\delta) > 0$ small enough,$\phi - L\phi \geq 0$ and $\phi = 1$ on the boundary. Therefore

$$|u(0)| \leq \phi(0) \sup_{y:|y|=\delta} |u(y)|$$

and we can take $\rho(\delta) = \exp[-c(\delta)]$. □

Lemma 9.6. *If u is a bounded solution of $u - Lu = 0$ outside some ball $|x| \geq r$ then for some $c > 0$ and $C < \infty$,*

$$\sup_{|x|=R} |u(x)| \leq Ce^{-c(R-r)} \sup_{|x|=r} |u(x)|$$

Proof. By the previous two lemmas

$$\sup_{|x|=r+\delta} |u(x)| \leq \rho(\delta) \sup_{|x|=r} |u(x)|$$

The lemma is now easily proved by induction. □

Suppose L is given and we modify L outside a ball $B(x_0, 4\delta)$ to get L' which has coeffecients $\{a'_{i,j}(x)\}$ that are uniformly close to some constant $c_{i,j}$ and $\{b'_j(x)\}$ are 0 in the complement of the ball. For δ small it is easy to see that our basic perturbation argument works for L' and

$$u - L'u = f$$

has a solution in $W_{2,p}$ for $f \in L_p$. In particular if $f \geq 0$ is supported inside $B(x_0, \delta)$, $L'u = u$ outside the ball and if $u \in W_{2,p}$, it is in some better L_{p_1} by Sobolev's lemma. We can iterate this process and obtain eventually an L_∞ bound of the form

$$\sup_{|x-x_0|=2\delta} |u(x)| \leq C\|f\|_p$$

If we now compare the solutions $u - L'u = f$ and $v - Lv = f$, both of which are nonnegative, since $L = L'$ inside $B(x_0, 4\delta)$, we have

$$(u - v) - L(u - v) = 0$$

Therefore

$$\sup_{|x-x_0|=2\delta} |u(x) - v(x)| \leq \rho(\delta) \sup_{|x-x_0|=4\delta} |u(x) - v(x)|$$

From this we conclude that

$$\sup_{|x-x_0|=2\delta} v(x) \leq \sup_{|x-x_0|=2\delta} u(x) + \rho(\delta)[\sup_{|x-x_0|=4\delta} u(x) + \sup_{|x-x_0|=4\delta} v(x)]$$

But

$$\sup_{|x-x_0|=4\delta} v(x) \leq \rho(\delta) \sup_{|x-x_0|=2\delta} v(x)$$

and

$$\sup_{|x-x_0|=4\delta} u(x) \leq \rho(\delta) \sup_{|x-x_0|=2\delta} u(x)$$

We see now that

$$\sup_{|x-x_0|=2\delta} v(x) \leq C(\delta) \sup_{|x-x_0|=2\delta} u(x) \leq C\|f\|_p$$

Now one can estimate $\|v\|_p \leq C\|f\|_p$.