10 Banach Alegebras, Wiener's Theorem

There is a theorem due to Wiener that asserts the following.

Theorem 10.1. *Suppose $f(x)$ on the d-torus has an absoutely convergent Fourier Series and $f(x)$ is nonzero on the d-torus. Then the function $g(x) = \frac{1}{f(x)}$ also has an absolutely convergent Fourier Series.*

Wiener's original proof involves direct estimation. We will give a "soft" proof using functional analysis techniques developed by Naimark. The proof will be broken up in to several steps as we develop the theory.

A (commutative) Banach algebra X is a Banach space with (associative) mutiplication of two elements u, v defined as uv satisfying $\|uv\| \leq \|u\|\|v\|$. It is commutatitve if $uv = vu$. A Banach algebra with a unit is one which has a special element called 1 such that $1u = u$ for all u. Such a unit is unique because if $1, 1'$ are two units then $11' = 1 = 1'$. An element u is invertible if there is a v such that $uv = 1$. The element v, which is unique if it exists, is called the inverse of u. The unit 1 is its own inverse. We can assume with out loss of generality that $\|1\| = 1$ by replacing $\|u\|$ by the equivalent norm of the operator $T_u : T_u v = uv$. An ideal I is a subspace with the property that whenever $x \in X, y \in I$ it follows that $xy \in I$. An ideal is proper if it is not X and not just the 0 element. A proper ideal can not contain 1 or any invertible element. A proper ideal is maximal if it is not contained in any other proper ideal. The closure of a proper ideal is proper. This needs proof. By a power series expansion if $\|1 - u\| < 1$, then u has an inverse $v = \sum_0^\infty (1-u)^n$. Therefore any proper ideal is disjoint from the open unit ball around 1. So does its closure. We can therefore assume that all our ideals are closed. Any ideal can be enlarged to a maximal ideal. Just apply Zorn's lemma and take the maximal element among those that do not intersect the unit ball around 1. If I is any (closed) ideal $X \backslash I$ is again a Banach Algebra, called the quotient algebra. If the ideal I is maximal then the quotient $Y = X \backslash I$ has no proper ideals. In such an algebra every nonzero element is invertible. Just look at the range of yY. It is an ideal. If $y \neq 0$, since it can not be proper, it must be all of Y, and therefore contains 1.

Theorem 10.2. *A Banach algebra with a unit and with out proper ideals over the complex numbers is the complex numbers.*

Proof. Since every non zero element is invertible, if there is an element u which is not a complex mutiple of 1, $(z1 - u)^{-1} = f(z)$ exists for all $z \in \mathbf{C}$

and is an entire function with values in Y.For $z > \|u\|$ we can represent $f(z) = \sum_{n\geq 0} \frac{u^n}{z^{n+1}}$. Therefore $\|f(z)\| \to 0$ as $z \to \infty$ and by Liouville's theorem must be identically zero. Contradiction. □

We now know that for any maximal ideal I, the map $X \to Y$ that sends $x \to x + I$ is a homomorphism onto $\mathbf{C}$.

Theorem 10.3. *If $u \in X$ is not invertible, then there is a homomorphism, i.e a mutiplicatve bounded linear functional, h such that $h(u) = 0$.*

Proof. Consider the ideal uX and enlarge it to a maximal ideal I, and then take the natuaral homomorphism into $\mathbf{C} = X \backslash I$. □

Corollary 10.1. *$u \in X$ is invertible if and only if $h(u) \neq 0$ for every homomorphism h.*

Consider the Banach algebra X of absolutely convergent Fourier Series $\sum_{n\in Z^d} a_n e^{i<n,x>}$ with norm $\sum_{n\in Z^d} |a_n|$. Wiener's theorem will be proved if we show

Theorem 10.4. *Every homomorphism h on X is given by*

$$h(u) = \sum_{n\in Z^d} a_n e^{i<n,\theta>}$$

for some θ on the d-torus.

Proof. Let $h(u_j) = z_j \in \mathbf{C}$ where $u_j = e^{ix_j}$. Since $\|u_j^k\| = 1$ for all positive and negative integrs k and h is a homomorphism z_j^k must be bounded and therefore $|z_j| = 1$. If we write $z_j = e^{i\theta_j}$, then $h(u) = \sum_{n\in Z^d} a_n e^{i<n,\theta>}$ for finite sums and since they are dense we are done. □