11 Compact Groups. Haar Measure.

A group is a set G with a binary operation $G \times G \to G$ called multiplication written as $gh \in G$ for $g, h \in G$. It is associative in the sense that $(gh)k = g(hk)$ for all $g, h, k \in G$. A group also has a special element e called the identity that satisfies $eg = ge = g$ for all $g \in G$. It is easy to verify that e is unique. A group also has the property that for each $g \in G$ there is an element $h = g^{-1}$ such that $gh = hg = e$. In general it need not be commutative i.e. $gh \neq hg$. If $gh = hg$ for all $g, h \in G$ the group is called abelian or commutative.

A topological group is a group with a topology such that the binary operation $G \times G \to G$ that sends $g, h \to gh^{-1}$ is continuous. This is seen to be equivalent to the assumption that the operations $G \times G \to G$ defined by $g, h \to gh$ and $G \to G$ defined by $g \to g^{-1}$ are continuous.

We will assume that our group G as a topological space is a compact metric space. Let $\mathcal{B}$ be the class of Borel sets. A measure on G is a nonnegative measure of total mass 1 on $(G, \mathcal{B})$. A left invariant (right invariant) Haar measure on G is a measure λ such that $\lambda(g^{-1}A) = \lambda(A)$ $(\lambda(Ag^{-1}) = \lambda(A))$ for all $A \in \mathcal{B}$. Here $g^{-1}A$ is the set of elemnts of the form $g^{-1}h$ with $h \in A$. The set Ag^{-1} is defined similarly.

For any two measures α, β on $(G, \mathcal{B})$ the measure $\alpha * \beta$ is defined by

$$\alpha * \beta(A) = \int_G \alpha(Ag^{-1}) d\beta(g) = \int_G \beta(g^{-1}A) d\alpha(g)$$

In terms of integrals

$$\int_G f(k) d(\alpha * \beta)(k) = \int_G \int_G f(gh) d\alpha(g) d\beta(h)$$

Theorem 11.1. *The following are equivalent.*

1. *λ is a left invariant Haar measure on G.*
2. *λ is a right invariant Haar measure on G.*
3. *λ is an idempotent i.e $\lambda * \lambda = \lambda$ and $\lambda(U) > 0$ for every open set U.*

The proof will depend on the following

Lemma 11.1. *$\alpha * \lambda = \lambda$ for an α with $\alpha(U) > 0$ for all open sets $U \subset G$ if and only if $\lambda(g^{-1}A) = \lambda(A)$ for all $g \in G$ i.e. λ is a left invariant Haar measure.*

Proof. For any bounded continuous function u on G let us define

$$v(g) = (u * \lambda)(g) = \int_G u(gh) d\lambda(h)$$

Then for any $a \in G$

$$\begin{aligned}
\int v(ag) d\alpha(g) &= \int_G \int_G u(agh) d\alpha(g) d\lambda(h) \\
&= \int_G u(ak) d(\alpha * \lambda)(k) \\
&= \int_G u(ak) d\lambda(k) \\
&= v(a)
\end{aligned}$$

We can take a to be the element where the maximum of v is attained. Then $v(ag) = v(a)$ for all g in the support of α. In particular $v(ag)$ and therefore v is a constant. Hence $\delta_g * \lambda = \lambda$ or λ is left invariant. $\square$

Proof. (of Theorem). If λ is right invariant then $\lambda * \delta_g = \lambda$ for all $g \in G$ and by integrating $\lambda * \lambda = \lambda$ and that implies that λ is left invariant as well. We note that any left or right invriant meausre cannot give zero mass to any open set because by compactness G can be covered by a finite number of translates of U. $\square$

Theorem 11.2. *A left (or right) invariant Haar meausre exists, is unique and is invariant from the right (left) as well.*

Proof. Let us start with any α that gives positive mass to every open set and consider

$$\lambda_n = \frac{\alpha + \alpha^2 + \alpha^n}{n}$$

Any weak limit λ of λ_n satisfies $\alpha * \lambda = \lambda$ and by lemma such a λ is left and therefore right invariant. If λ is left invariant then $\alpha * \lambda = \lambda$ for any α and if α is also right invariant then $\alpha * \lambda = \alpha$ proving uniqueness. $\square$

12 Representations of a Group

Given a group G, a representation $\pi(g)$ of the group is a continuous mapping $\pi(\cdot)$ of G into nonsingular linear transformations of a finite dimensional complex vector space V such that $\pi(e) = I$ and $\pi(gh) = \pi(g)\pi(h)$ for all $g, h \in G$.

Theorem 12.1. *Given a representation π of G on a finite dimensional vector space V, there is an inner product $< v_1, v_2 >$ on V, such that each $\pi(g)$ is a unitary transformation.*

Proof. Let $< v_1, v_2 >_0$ be any inner product. We define a new inner product

$$< v_1, v_2 >= \int_G < \pi(h)v_1, \pi(h)v_2 >_0 \, dh$$

where dh is the unique Haar measure on G. It is seen that

$$\begin{aligned} < \pi(g)v_1, \pi(g)v_2 > &= \int_G < \pi(h)\pi(g)v_1, \pi(h)\pi(g)v_2 >_0 \, dh \\ &= \int_G < \pi(hg)v_1, \pi(hg)v_2 >_0 \, dh \\ &= \int_G < \pi(h)v_1, \pi(h)v_2 >_0 \, dh \\ &=< v_1, v_2 > \end{aligned}$$

which proves that $\pi(\cdot)$ are unitary with respect to $< \cdot, \cdot >$. ☐

A representation $\pi(\cdot)$ of G on V is irreducible if there is no proper subspace W of U other than U itself and the subspace $\{0\}$ that is left invariant by $\{\pi(g) : g \in G\}$. Since any finite dimensional representation of V can be made unitary, if $W \subset U$ is invariant so is $W^\perp$ and $\pi(\cdot)$ on V is the direct sum of $\pi(\cdot)$ on W and $W^\perp$. It is clear that any finite dmensional representation is a direct sum of irreducible representations.

Two unitary representations π_1 and π_2 of G on two vector spaces V_1 and V_2 are said to be equivalent if there is a linear isomorphsim $T : V_1 \to V_2$ such that $\pi_2(g)T = T\pi_1(g)$ for all $g \in G$. It is clear that π_1 and π_2 are equivalent then either they are both irreducble or neither is. The set of irreducible representations is naturally divided into equivalence classes. We denote them by $\omega \in \Omega$. Each ω is an equivalence class and Ω is the set of all equivalence classes.

Lemma 12.1. *If π is an irreducible representation of G on a finite dimensional vector space V, then the inner product that makes the representation unitary is unique upto a scalar multiple.*

Proof. If $< \cdot, \cdot >_i$: $i = 1, 2$ are two inner products on V that make $\pi(g)$ unitary for all $g \in G$, then

$$< \pi(g)u, \pi(g)v >_i = < u, v >_i$$

and if we represent by T any unitary isomorphism between the two inner product spaces $\{V, < \cdot, \cdot >_1\}$ and $\{V, < \cdot, \cdot >_2\}$ so that $< u, v >_1 = < Tu, Tv >_2$ then

$$< Tu, Tv >_2 = < u, v >_1 = < \pi(g)u, \pi(g)v >_1 = < T\pi(g)u, T\pi(g)v >_2$$

In other words if we denote by T^* the adjoint of T, on the inner product space $< \cdot, \cdot >_2$

$$(T^*T)\pi(g) \equiv \pi(g)(T^*T)$$

for all $g \in G$. T^*T is Hermitian and its eigenspaces are left invariant by the i$\pi(g)$ that commute wth it. These eigenspaces have to be trivial because of rreducibility. That forces T^*T to be a positive mutiple of identity. The two inner products are then essentially the same. ∎

Given two representations π_1 and π_2 on V_1 and V_2 an intertwining operator from $V_1 \to V_2$ is a linear map T such that $T\pi_1(g) = \pi_2(g)T$ for all $g \in G$.

Theorem 12.2. Schur's Lemma. *Given two irreducible representaions π_i on V_i any intertwining operator T, it is eiher an isomorphism which makes the two representations equivalent or $T = 0$.*

Proof. Suppose T has a null space $W \subset V_1$. Then if $u \in W$, $T\pi_1(g)u = \pi_2(g)Tu = 0$ so that $\pi(g)W \subset W$. Either $W = \{0\}$ or $W = U_1$ making W either 0 or one-to-one. By a similar argument the range of T is left invariant by π_2 making T either 0 or onto. Therefore if T is not an isomorpihsm it is 0. Any isomorphism is essentially a unitary isomorphism. ∎

13 Representations of a Compact group.

A natural infinite dimensional representation of a compact group is the (left) regular representation L_g on $L_2(G, dg)$ defined by $(L_g u)(h) = u(g^{-1}h)$ satisfying $L_{g_1} L_{g_2} = L_{g_1 g_2}$. From the invariance of the Haar measure L_g is unitary.

First we prove some facts regarding finite dimensional representations of compact groups.

Theorem 13.1. *Let π be a finite dimnsional irreducible representation of G in a space of dimension d. Then there is subspace of dimension d^2 in $L_2(G, dg)$ that is invariant under both left and right regular representations and either one on this subspace decomposes into d copies of π. The representation of this type, i.e. any equivalent representation does not occur in the orthogonal complement of this d^2 dimensional subspace.*

Proof. Let π be a representation of G in a finite dimensional space V. Pick a basis for V and represent $\pi(g)$ as a unitary matrix $\{t_{i,j}(g)\}$. The functions $t_{i,j}(\cdot)$ are continuous and are in $L_2(G, dg)$. Let us see what the left regular representation L_h does to them.

$$t_{i,j}(hg) = [\pi(h)\pi(g)]_{i,j} = \sum_r t_{i,r}(h) t_{r,j}(g)$$

which is the same as

$$L_h t_{i,j}(\cdot) = \sum_r t_{i,r}(h) t_{r,j}(\cdot)$$

or for each j the space spanned by $\{t_{r,j}(\cdot) : 1 \le r \le d\}$ is invariant under L_h and transforms like π. Similarly under $R_{h^{-1}}$, the rows $\{t_{j,r}(\cdot) : 1 \le r \le d\}$ will again transform like π. If we can show that $\{t_{i,j}(\cdot)\}$ are linearly independent then d^2 dimensional space will transform like d copies of π under L_h and R_h. Consider two representations π_1,π_2 that are irreducible on V_1, V_2 and vectors u_1, v_1 and u_2, v_2 in their respective spaces. Then for fixed v_1, v_2

$$\int_G < \pi_1(g)u_1, v_1 >_1 \overline{< \pi_2(g)u_2, v_2 >}_2 dg = < Bu_1, u_2 >$$

defines an operator from $V_1 \to V_2$ and a calculation

$$\begin{aligned}
< B\pi_1(h)u_1, u_2 > &= \int_G < \pi_1(g)\pi_1(h)u_1, v_1 >_1 \overline{< \pi_2(g)u_2, v_2 >_2} dg \\
&= \int_G < \pi_1(gh)u_1, v_1 >_1 \overline{< \pi_2(g)u_2, v_2 >_2} dg \\
&= \int_G < \pi_1(g)u_1, v_1 >_1 \overline{< \pi_2(gh^{-1})u_2, v_2 >_2} dg \\
&= \int_G < \pi_1(g)u_1, v_1 >_1 \overline{< \pi_2(g)\pi_2(h^{-1})u_2, v_2 >_2} dg \\
&= \int_G < \pi_1(g)u_1, v_1 >_1 \overline{< \pi_2(g)\pi_2^*(h)u_2, v_2 >_2} dg \\
&= < Bu_1, \pi_2^*(h)u_2 > \\
&= < \pi_2(h)Bu_1, u_2 >
\end{aligned}$$

Showing that B intertwines π_1 and π_2. If the representations are inequivalent then $B = 0$. Otherwise, $B = c(v_1, v_2)T$ where T is the isomorphism between V_1 and V_2 that intertwines π_1 and π_2 and a further calculation of the same nature shows that $c(v_1, v_2) = c < Tv_1, v_2 >_2$. Therefore

$$\int_G < \pi_1(g)u_1, v_1 >_1 \overline{< \pi_2(g)u_2, v_2 >_2} dg = c < Tu_1, u_2 >_2 < Tv_1, v_2 >_2$$

where $c = 0$ for inequivalent representations. In the equivalent case taking $\pi \equiv \pi_1 \equiv \pi_2$, $V = V_1 = V_2$ and $T = I$,

$$\int_G < \pi(g)u_1, v_1 > \overline{< \pi(g)u_2, v_2 >} dg = c < u_1, u_2 >_2 < v_1, v_2 >$$

To calculate c we take $u = u_1 = u_2$ and $v = v_1 = v_2$

$$c\|u\|^2\|v\|^2 = \int_G | < \pi(g)u, v > |^2 dg = \int_G |\sum_{i,j} t_{i,j}(g)u_i v_j|^2 dg$$

Thus

$$\int_G t_{i,j}(g)\overline{t_{rs}(g)} dg = c\delta_{ir}\delta_{js}$$

On the other hand

$$\sum_{i,j} |t_{i,j}|^2 \equiv d$$

so that $c = \frac{1}{d}$. The character of the represenation defined as $\chi_\pi(g) = \text{trace}\, \pi(g)$ is independent of concrete vector space used for the representation and $\|\chi_\pi\|_{L_2(G,dg)} = 1$. If π occured again in the orthogonal complement of the d dimensional space, then that space would have to contain χ_π again and that is not possible. $\square$

Now we show that there are lots of finite dimensional representations.

Theorem 13.2. *Any right trnaslation R_k defined by $(R_k u)(h) = u(hk)$ commutes with L_g. There are compact self adjoint operators that commute with the family $\{L_g\}$. Hence the representation L_g on $L_2(G, dg)$ decomposes into a sum of irreducible finite dimensional represenations. In particular a compact group has sufficiently many irreducible finite dimensional representations. More precisely given $g \neq e$ there is one for which $\pi(g) \neq I$.*

Proof. Clearly
$$(R_k L_g u)(h) = (R_k L_g u)(h) = u(g^{-1}hk)$$
The integral operators

$$(Tu)(h) = \int_G u(hg)\tau(g)dg = \int_G u(g)\tau(h^{-1}g)dg$$

commute with L_g and are compact (in fact Hilbert-Schmidt) and self adjoint provided
$$\int_G \int_G |\tau(h^{-1}g)|^2 dg dh < \infty$$
and for all k,
$$\tau(k^{-1}) = \overline{\tau(k)}$$
The eigenspaces of T of finite dimension provide lots of finite dimensional representations that can then be split up into irreducible pieces. We can check that the only possible infinite dimensional piece is the null space of T. Let us write $L_2(G, dg) = \oplus_j V_j \oplus V_\infty$ as the direct sum of finite dimensional pieces that are invariant under both L_g and R_g if possible an infinite dimensional piece V_∞ that is also invariant under L_g and R_g and has no such nontrivial finite dimensional invariant subspace. The earlier argument of convolution by τ can be repeated on V_∞ and produces a finite dimensional L_g invariant subspace, which is a contradiction unless such a convolution is identically 0 on V_∞ for all τ. This is seen to be impossible. $\square$

The character $\chi_\pi(g)$ determine π completely and for ineuqivalent representations they are orthogonal. The cahracters have the additonal property that $\chi_\pi(hgh^{-1}) = \chi_\pi(g)$.

Theorem 13.3. *Any function $u(g) \in L_2(G, dg)$ such that $u(hgh^{-1}) = u(g)$ i.e. $L_h u = R_h u$ for all $h \in G$ is spanned by the characters.*

Proof. We need to show that if any such u is orthogonal to χ_π it is also orthogonal to all the matrix elements.

$$\begin{aligned}
\int_G u(g)\overline{t_{r,s}(g)}dg &= \int u(hgh^{-1})\overline{t_{r,s}(g)}dg \\
&= \int_G u(g)\overline{t_{r,s}(h^{-1}gh)}dg \\
&= \int_G u(g)\overline{[\pi^*(h)\pi(g)\pi(h)]_{r,s}}dg \\
&= \int_G u(g)\sum_{i,j} t_{i,r}(h)\overline{t_{i,j}(g)t_{j,s}(h)}dg \\
&= \int_G\int_G u(g)\overline{t_{i,j}(g)}t_{i,r}(h)\overline{t_{j,s}(h)}dgdh \\
&= \sum_{i,j}\frac{1}{d}\delta_{i,j}\delta_{r,s}\int_G u(g)\overline{t_{i,j}(g)}dg \\
&= \frac{1}{d}\delta_{r,s}\int_G u(g)\overline{\chi_\pi(g)}dg = 0
\end{aligned}$$

□

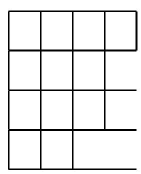
14 Representations of the permutation group.

Permutations on n symbols is the set of one to one mappings σ of a set $X = \{x_1, x_2, \ldots, x_n\}$ of n elemnts on to itself. It is a finite group G with $n!$ elements. Given $\sigma \in G$ we can look at the orbits of $\sigma^n x$ and it will partition the speace X into orbits $A_1, A_2, \ldots, A_k$ consisting of $n_1 \geq \ldots \geq n_k$ points so that $n = n_1 + \cdots + n_k$. If $\hat\sigma = s\sigma s^{-1}$ is conjugate to σ then the orbits of $\hat\sigma$ will be $sA_1, sA_2, \ldots, sA_k$ so that the partition $n_1 \geq \ldots \geq n_k$ of n into k numbers will be the same for σ and $\hat\sigma$. Conversely if σ and $\hat\sigma$ has the same partition of n, then the orbits $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ have the same cardinalities.

We can find $s \in G$ that maps $A_i \to B_i$ in a one-to-one and onto manner and we can reduce the problem to the case where $A_i = B_i$ for every i. We can now relable the points with in each A_i i.e. find a permutation of n_i elements, such that bothe σ and $\hat{\sigma}$ look the same on each A_i. We have therefore proved

Theorem 14.1. *The number of distinct inequivalent irreducible representations of G is the same as the number of distinct partitions $\mathcal{P}(n)$ of the integer n.*

Just note that they are both equal to the dimension of the subspace $\mathcal{X}$ of functions u satisfying $u(h^{-1}gh) = u(g)$ for all $h, g \in G$. We know that the characters $\chi_\pi(g) = \text{trace } \pi(g)$ of all the equivalence classes of representations span $\mathcal{X}$ so that the number of such equivalence classes is also $\mathcal{P}(n)$. We will now construct a distinct representation for each distinct partition of n. Given a partition λ of n into $n_1 \geq n_2 \geq \cdots \geq n_k$, we associate a diagram called the Young diagram corresponding to λ. It looks like



when $n = 12$ and the partition is $4, 3, 3, 2$. A Young tableau t is a diagram λ with the boxes filled in arbitrarily by the numbers $1, 2, \cdots, 8$, like

7	4	1
6	5	
8	3	
2		

The rows of the array are of length $n_1 \geq \cdots \geq n_k$. For any diagram there are $n!$ tableaux. A tabloid is when the order of entries are not relevent. The tabloid $\{t\}$ consists of $\{7, 4, 1\}, \{6, 5\}, \{8, 3\}, \{2\}$. There are $\frac{n!}{n_1! \cdots n_k!}$ tabloids corresponding to any tableau t of the diagram λ. The subgroup C_t of the permutaion group consists of permutaions within each column. In our case it consists of 72 elements an arbitray permutation of $7, 6, 8, 2$ and an arbitrary permutation of $4, 5, 3$. For any permutaion s, $\sigma(s) = \pm 1$ is the sign of the permutation. We define an abstract inner product space V of dimension $\frac{n!}{n_1! \cdots n_k!}$ with the orthonormal basis $e_{\{t\}}$ as $\{t\}$ varies over the tabloids of the tableau t corresponding to λ. We define $n!$ vectors e_t in V by

$$e_t = \sum_{s \in C_t} \sigma(s) s e_{\{t\}}$$

The e_t may not be linearly independent and the span of $\{e_t\}$ is denoted by W. One defines a representation of $\pi_\lambda(g)$ of the permutation group on W corresponding to the diagram λ by defining

$$\pi(g) e_t = g e_t$$

Theorem 14.2. *Each π_λ is irreducible. For two distinct diagrams they are inequivalent. We therefore have all the representations.*

Proof is broken up into lemmas.

Lemma 14.1.

$$\begin{aligned}\sum_{s \in C_t} \sigma(s) g s e_{\{t\}} = \sum_{s \in C_t} \sigma(s) g s g^{-1} g e_{\{t\}} &= \sum_{g s g^{-1} \in C_{gt}} \sigma(s) g s g^{-1} g e_{\{t\}} \\ = \sum_{s \in C_{gt}} \sigma(s) s e_{\{gt\}} = e_{gt}&\end{aligned}$$

Lemma 14.2. *Suppose λ, μ are two different diagrams t a λ-tableau and τ a μ-tableau. Suppose*

$$\sum_{s \in C_t} \sigma(s) e_{\{s\tau\}} \neq 0$$

Then $n_1 \geq m_1, n_1 + n_2 \geq m_1 + m_2, \cdots$ where $n_1, \ldots, n_k$ and $m_1, \ldots, m_\ell$ are the two partitions corresponding to λ and μ. We say then that $\lambda \geq_1 \mu$. If $\lambda = \mu$ then the sum is $\pm e_t$.

Proof. Suppose that two elements a, b are in the same row of τ and in the same column of t. Then the permutation $p = \{a \leftrightarrow b\}$ is in C_t, $spe_{\{t\}} = se_{\{t\}}$, with $\sigma(sp) + \sigma(s) = 0$. So the sum adds up to 0 which is ruled out. Hence, no two elements in the same row of τ can be in the same column of t. In particular t must have atleast as many columns as the number of elements in the first row of τ proving $n_1 \geq m_1$. A variant of this argument proves $\lambda \geq_1 \mu$. Suppose now that $\lambda = \mu$. Again all the elments in any row of τ appear in different columns of t. So there is a permutation $s^* \in C_t$ such that $s^* t = \tau$. The sum is unaltered if we replace τ by $s^* t$ except for $\sigma(s^*) = \pm 1$. $\square$

Lemma 14.3. *Let $u \in W$ corresponding to a diagram μ. Let t be any μ-tableau. Then*

$$\sum_{s \in C_t} \sigma(s)\pi(s)u = ce_t$$

Proof. u is a linear combination of e_τ for different λ-tableau τ. Each one from the previous lemma yields ce_τ with $c = 0, \pm 1$. Add them up! $\square$

Let us define

$$A_t = \sum_{s \in C_t} \sigma(s)\pi(s)$$

We alraedy have an inner product that makes $\pi(s)$ orthogonal.

$$\begin{aligned} < A_t u, v > &= \sum_{s \in C_t} \sigma(s) < \pi(s)u, v > \\ &= \sum_{s \in C_t} \sigma(s) < u, \pi(s^{-1})v > \\ &= \sum_{s \in C_t} \sigma(s) < u, \pi(s)v > \\ &=< u, A_t v > \end{aligned}$$

because $sigma(s^{-1}) = \sigma(s)$.

Lemma 14.4. *If U is any invariant subspace of V then either $U \supset W$ or $U \perp W$. This proves the irreducibility of W.*

Proof. Suppose $u \in W$ and t is a λ-tableau. We saw that $A_t u = c_t e_t$ for some constant c_t. Suppose for some t, $c_t \neq 0$. Then $e_t \in U$ and hence $W \subset U$. If $c_t = 0$ for all t, $0 =< A_t u, e_{\{t\}} >=< u, A_t e_{\{t\}} >=< u, e_t >$ and $u \in W^\perp$. $\square$

Lemma 14.5. *Let T intertwine the representations on V^λ and V^μ. Suppose W^λ is not contained in* Ker T*. Then $\lambda \geq_1 \mu$.*

Proof. Ker$T$ is invariant under $\pi(g)$ and if it does not contain W^λ it is orthogonal to it.

$$0 \neq Te_t = TA_t e_{\{t\}} = A_t T e_{\{t\}}$$

$Te_{\{t\}}$ is a combination of $e_{\{\tau\}}$ of μ-tableaux τ. So atleast one of them $A_t e_{\{\tau\}}$ is nonzero forcing $\lambda \geq_1 \mu$. $\square$

Lemma 14.6. *If $T \neq 0$ intertwines W^λ and W^μ, then $\lambda = \mu$.*

Proof. Extend T by making it 0 on $(W^\lambda)^\perp$ and we see that $\lambda \geq_1 \mu$. The argument is symmetric. $\square$