Lecture 1.(Jan 19, 2000)

1. Construct explicitly a continuous function f on S, such that the Fourier Series of f does not converge uniformly.

2. Show that the Fourier Series of any f, which is Hölder continuous with some exponent $\alpha > 0$, converges uniformly.

Lecture 2.(Jan 26, 2000)

1. For the Fejer kernel

$$S_N(f, x) = \frac{1}{2\pi} \int f(x - y) K_N(y) dy$$

prove the maximal inequality

$$\mu(x : \sup_{N \geq 1} |S_N(f, x)| \geq \ell) \leq \frac{C \|f\|_1}{\ell}$$

2. State and prove a reasonable multidimensional analog of the Hardy-Littlewood maximal inequality. If the maximal function is defined as

$$M_f(x) = \sup_{R \in \mathbf{R}_x} \frac{1}{\mu(R)} \int_R |f(y)| dy$$

where $\mathbf{R}_x$ is the class of all rectangles with sides parallel to the axes that have x as center, is a weak type $(1, 1)$ inequality valid? What if we allow arbitrary orientation ?

Lecture 3. Feb 2,2000

Lacunary Series: Fourier Series of the form

$$\sum_{k \geq 1} a_k \cos n_k x$$

(for example with $n_k = 2^{2^k}$) so that

$$n_{k+1} \geq k(n_1 + n_2 + \ldots + n_k)$$

for every k are called lacunary series. They provide good counter examples. Such a series behaves like series of independent random variables on the probability space $[-\pi, \pi]$ with the normalized Lebesgue measure $d\mu = \frac{dx}{2\pi}$.

1. On the probability space of $[-\pi, \pi]$ with normalized Lebesgue measure $\frac{dx}{2\pi}$ prove the central limit theorem for the random variables

$$S_N = \sqrt{\frac{2}{N}} \sum_{j=1}^{N} \cos n_j x$$

by calculating the moments and showing that for every $k \geq 1$,

$$\lim_{N\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} [S_N(x)]^k dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^k \exp[-\frac{y^2}{2}] dy$$

2. Construct a sequence of functions

$$f_k(x) = \sum_n a_{k,n} e^{i\,n\,x}$$

such that

$$\lim_{k\to\infty} \sum_n |a_{k,n}|^p = 0$$

for every $p > 2$ while for every $\ell > 0$

$$\lim_{k\to\infty} \mu[x : |f_k(x)| \geq \ell] = 1$$

so that f_k does not go to 0, in any reasonable space of functions.

Lecture 4, Feb 9, 2000

1. Suppose we have in the plane, a function $K(x, y)$ of the form $\frac{K(\theta)}{r^2}$ in polar coordinates. Assume that $K(\theta)$ is a nice periodic function of period 2π that has mean 0, i.e $\int_0^{2\pi} K(\theta) d\theta = 0$. Compute its Fourier transform

$$k(\xi, \eta) = \int_{R^2} \exp[i(\xi x + \eta y)] K(x, y) dx dy$$

and show that it is a homogeneous function of degree 0.

2. Consider the following class of operators on R^2

$$\widehat{Tf} = k(\theta) \hat{f}$$

where $\hat{f}$ is the fourier transform of f and θ is the angle in the polar coordinates (r, θ). Find a representation for T, as a convolution with a kernel $K(x, y)$ of the type considered in problem 1. Find reasonable sufficient conditions on k, under which T is a bounded operator from L_p to L_p for $1 < p < \infty$.

3. Is there a generalization to R^d for $d > 2$?

Lecture 4 Feb 16, 2000.

Q 1. Let $u \in W_{k,p}$ for some positive integer k and $1 < p < \infty$. if we define the translations $T_{i,h}$ by

$$T_{i,h}u = u(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_d)$$

show that the limits of difference quotients

$$D_{x_i}u = \lim_{h \to 0} \frac{1}{h}[T_{i,h}u - u]$$

exist in $W_{k-1,p}$ and define a bounded operator D_{x_i} from $W_{k,p}$, into $W_{k-1,p}$

Q 2. Let $d = 1$ and $u(x) \in W_{1,1}$ Show that u is continuous at every x and differentiable in the usual sense at almost all x, i.e for almost all x,

$$\lim_{h \to 0} \frac{u(x+h) - u(x)}{h} = (Du)(x)$$

Lectures 5-6 March 1, 2000

Q 1. Suppose f is given by a rational function

$$f(e^{i\theta}) = \frac{|P(e^{i\theta})|^2}{|Q(e^{i\theta})|^2}$$

where P and Q are polynomials and Q has no zeros on the unit circle. Calculate the projection of 1 on the span of $\{e^{ik\theta} : k \geq 1\}$ and the projection error.

Q 2. If

$$\int_0^{2\pi} \log f(\theta) d\theta = -\infty$$

show that $H_k = H_{k+1}$ for every k.

Q 3. Can you show that if $\mu \perp d\theta$ then again $H_k = H_{k+1}$ for all k?

Lectures 7 March 8, 2000

Q 1. Show that the function $\log|x-y|$ is a BMO function of x for each y.

Q 2. What about

$$U(x) = \int \log|x-y|d\mu(y)$$

for some finite measure μ.

Q 3. A function $u(x)$ is said to be in the class VMO (Vanishing mean oscillation) if

$$\lim_{h\to 0} \sup_{Q:|Q|\leq h} \frac{1}{|Q|}\int_Q |u(x)-u_Q|dx = 0$$

Can you find a function that is BMO but not VMO?

Can you find a function that is VMO but not continuous?