4.6 Example of non-uniqueness.

If we try to construct a solution to the martingale problem in 1 dimension coresponding to $a(x) = |x|^\alpha$ with $0 < \alpha < 1$, it is easy to show nonuniqueness, due to the nature of the vanishing of $a(x)$ near 0. In particular, if we start at time 0 from the point 0, because $a(0) = 0$, the measure P such that $P[x(t) \equiv 0] = 1$ is a solution. On the other hand we van try to get another solution by a random time chane from Brownian motion. We try to define P as the distribution of $\beta(\tau_t)$ where τ_t is the solution of

$$\int_0^{\tau_t} \frac{ds}{a(\beta(s))} = t.$$

To make sure that this is well defined, we must check that

$$\int_0^t \frac{ds}{|\beta(s)|^\alpha} < \infty \quad \text{a.e.}$$

We can use Fubini's theorem and check

$$E\left[\int_0^t \frac{ds}{|\beta(s)|^\alpha}\right] = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi s}} e^{-\frac{y^2}{2s}} \frac{dy}{|y|^\alpha} ds < \infty$$

Essentially the vanishing of $a(0)$ and the integrability of $\frac{1}{a(x)}$ near 0 cause the trouble. Compare it to the standard example of nonuniqueness for $\dot{x} = b(x)$ which arises from the vanishing, $b(0) = 0$ of b at 0, in such a way that $\int \frac{dx}{b(x)}$ remains integrable.

4.7 Higher dimensions.

The homogeneous or time independent case is special in $d = 2$. We want to prove existence and uniquness for $[a, 0]$ where

$$a(x,y) = \begin{pmatrix} a_{11}(x,y) & a_{12}(x,y) \\ a_{12}(x,y) & a_{22}(x,y) \end{pmatrix}$$

We can always do a random time change. If the matrix is uniformly elliptic, i.e., if $c_1 I \le a(x,y) \le c_2 I$, we can mutiply by a scalar function and normalize so that $\text{Trace} a(x,y) = a_{11}(x,y) + a_{22}(x,y) \equiv 2$. Let us assume with out loss of generality that this is indeed the case. Let $1 - a_{11}(x,y) = -(1 - a_{22}(x,y)) = \epsilon(x,y)$. Consider the solution of

$$\lambda u - \frac{\Delta}{2} u = f$$

for $f \in L_2(R^d)$. We will estimate the difference

$$\begin{aligned} \lambda u - \mathcal{L}u - f = g &= (\frac{\Delta}{2} - \mathcal{L})u \\ &= \frac{1}{2}[(1 - a_{11}(x,y))u_{xx} - 2a_{12}(x,y)u_{xy} + (1 - a_{22}(x,y))u_{yy}] \\ &= \frac{1}{2}[\epsilon(x,y)(u_{xx} - u_{yy}) - 2a_{12}(x,y)u_{xy}] \end{aligned}$$

$$|g|^2 \le \frac{1}{4}[\epsilon^2(x,y) + a_{12}^2(x,y)][(u_{xx} - u_{yy})^2 + 4u_{xy}^2]$$

If we denote by $\delta = \sup_{x,y}[\epsilon^2(x,y) + a_{12}^2(x,y)]$, and $\widehat{f}, \widehat{u}$ the Fourier transforms of f and u respectively

$$\|g\|_2^2 \le \frac{\delta}{4}[\|(\xi^2 - \eta^2)\widehat{u}\|_2^2 + 4\|\xi\eta\widehat{u}\|_2^2 = \frac{\delta}{4}\|(\xi^2 + \eta^2)\widehat{u}\|^2 \le \delta\|\widehat{f}\|_2^2 = \delta\|f\|_2^2$$

Moreover

$$\begin{aligned}
\epsilon^2(x,y) + a_{12}^2(x,y) &= -\det(a - I) \\
&= -(\lambda_1(x,y) - 1)(\lambda_2(x,y) - 1) \\
&= 1 - \lambda_1(x,y)\lambda_2(x,y) \\
&\le \delta < 1
\end{aligned}$$

because of ellipticity, where λ_i are the eigenvalues of a satisfying $\lambda_1(x,y) + \lambda_2(x,y) \equiv 2$ and $\lambda_i(x,y) \ge c_1$.

Exercise 4.1. Now follow the same proof as in the one dimensional case, except limit yourself to the time homogeneous case. The quantities

$$E^{P_x}\left[\int_0^\infty e^{-\lambda t} f(x(t))dt\right]$$

are detremined uniquely and through them the solution to the martingale problem as well.

Remark 4.13. The situation of the general time dependent elliptic case in $d \ge 2$ or even the time homogeneous case in $d \ge 3$ is more complex. Even for Brownian motion, objects of the form

$$\lambda(f) = E^x\left[\int_0^T f(t, x(t))dt\right]$$

or

$$\lambda(f) = E^x\left[\int_0^\infty e^{-\lambda t} f(x(t))dt\right]$$

are not bounded linear functionals of $f \in L_2([0,T] \times R^d)$ or $L_2(R^d)$ as the case may be. So the perturbation theory in L_2 does not work. However

$$\lambda(f) = E^x\left[\int_0^T f(t, x(t))dt\right]$$

is a bounded linear functional on $L_p([0,T] \times R^d)$ for some $p = p(d)$ that dpends on the dimension. There is a result in singular integrals that establishes that the operators

$$\frac{\partial^2}{\partial x_i \partial x_j}\int_s^T \int_{R^d} f(t,y)\frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{\|y-x\|^2}{2(t-s)}} dtdy$$

are bounded by some constant $C = C(p, d)$ from $L_p([0,T] \times R^d$ into itself. This allows the perturbation theory to work, but only if $\|a(t,x) - I\| \le \epsilon = \epsilon(p, \delta)$, where ϵ depends on d and the $p = p(d)$ that we have to use. All we can say that $\epsilon > 0$. The same applies if we try to perturb from any constant positive definite symmetric matrix $C = \{c_{i,j}\}$. The perturbation range depends only on the sizes of the smallest and largest eigenvalues of C.

Remark 4.14. The previous remark will enable us to prove the existence as well as uniqueness of solutions to the martingale problem for $[a, 0]$ where $a = a(t,x)$ is uniformly close to a constant positive definite matrix C. How close will depend on the dimension d as well as the upper and lower bounds on the eigen values of C. This is not very satisfactory. However this is enough to show that if $a(t,x)$ is uniformly bounded, continuous and nondegenerate for every (t,x) then we do have existence and uniqueness. Existence is done as usual by approximating by smooth coefficients, observing that the measures or totally bounded and extracting a convergent subsequence. Let us establish uniqueness. Assume that we have two solutions for the same $[a, 0]$ starting from (s_0, x_0). Let ϵ be the perturbation range that will work for $C = a(s,x)$ as (s,x) varies over a compact set $[0,T] \times \{x : |x| \le \ell\}$. If τ_1 is the exit time from the space-time neighborhood of size ϵ, then the processes do not 'know' that the coefficients are not with in ϵ of $C = a(s_0, x_0)$ and therefore any two solutions P_1 and P_2 have to agree on $\mathcal{F}_{\tau_1}$. Then a conditioning argument can be used to prove that the conditional distributions have to agree upto exiting from a space-time neighborhood of size ϵ from the first exit point $(\tau_1, x(\tau_1))$. Let us call this the second exit point $(\tau_2, x(\tau_2))$. Since the marginals and conditional determine the joint distribution, we have the two measures agreeing on $\mathcal{F}_{\tau_2}$. By induction they agree on $\mathcal{F}_{\tau_n}$. We let $n \to \infty$. Since we are limited to $[0,T] \times \{x : |x| \le \ell\}$ we can get P_1 and P_2 agreeing on $\mathcal{F}_{\sigma_\ell}$ where σ_ℓ is the exit time from $[0,T] \times \{x : |x| \le \ell\}$. Letting ℓ go to infinity we are done.

Remark 4.15. No matter how existence or uniqueness is proved, so long as a is nondegenerate with uniform upper and lower bounds on the eigenvalues we can always go from $[a, 0]$ to $[a, b]$ by Girsanov's formula provided $b = b(t,x)$ is bounded. Actually a stopping argument, that uses exit times from bounded sets can be employed and we can get away with assuming uniform nondegeneracy only on compact sets.

4.8 Convergence of Markov Chains.

Suppose for each $0 < h \le 1$, we are given the transition probability $\pi_h(x, dy)$ of a Markov Process on R^d. We think of h as the unit of time step and construct a measure $P_{h,x}$ on the space of sequences $\{x_n\}$ with values in R^d. The measure $P_{h,x}$ has the property that $P_{h,x}[x_0 = x] = 1$ and $\{x_n\}$ is a Markov Process under $P_{h,x}$ with $\pi_h(x, dy)$ as transition probability. We can tranfer the measure to the function space $\Omega = C[[0,\infty); R^d]$ by mapping $x(nh) = x_n$ and interpolating linearly in between to make it continuous. We will also denote by $P_{h,x}$ the measure on Ω . We are interested in the behavior of $P_{h,x}$ as $h \to 0$. It is natural to assume that

$$\lim_{h\to 0} \frac{1}{h} \int_{R^d} [f(y) - f(x)]\pi_h(x, dy) = (\mathcal{A}f)(x)$$

exists uniformly for x in compact subsets $K \subset R^d$ and for C^∞ functions f with compact support in R^d. The limit is of the form

$$(\mathcal{A}f)(x) = \frac{1}{2} \sum a_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_j b_j(x) \frac{\partial f}{\partial x_j}(x)$$

where $a = \{a_{i,j}(x)\}$ is a symmetric positive semidefinite matrix of diffusion coefficients and $b = b_j(x)$ are continuous drift coefficients. Let us assume that $[a, b]$ are uniformy bounded. Let us suppose that the solution to the martingale problem for $\mathcal{A}$ is unique for $[a, b]$, giving us a Markov family P_x of measures on Ω for $\mathcal{A}$. Our goal is to prove the theorem

Theorem 4.9. *As $h \to 0$, the family $\{P_{h,x}\}$ converges to $\{P_x\}$.*

Proof.
Step 1. Because we have uniform convergence only on compact subsets, it is better to introduce cutoff function $\phi_\ell(x) = \phi(\frac{x}{\ell})$ where $\phi(x)$ is smooth $0 \le \phi(x) \le 1$ with $\phi(x) = 1$ on $\{x : |x| \le 1\}$ and $\phi(x) = 0$ on $\{x : |x| \ge 1$. We then define

$$\pi_h^\ell(x, dy) = \phi_\ell(x)\pi_h(x, dy) + (1 - \phi_\ell(x))\delta(x, dy)$$

and the corresponding chains and measures $\{P_{h,x}^\ell\}$ on Ω.

Step 2. Relative to $(\Omega, \mathcal{F}_{nh}, P_{h,x}^\ell)$

$$Z_f^h(nh, \omega) = f(x(nh)) - f(x(0)) - \sum_{j=0}^{n-1} \int_{R^d} [f(y) - f(x)]\pi_{h,x}^\ell(x, dy)$$

is a martingale. Suppose that $\{P_{h,x}^\ell : h > 0\}$ is totally bounded. Then it is seen easily that

$$\lim_{\substack{h\to 0\\ nh\to t}} Z_f^h(nh, \omega) = f(x(t)) - f(x(0)) - \int_0^t \phi_\ell(x)(\mathcal{A}f)(x(s))ds$$

which implies that any limit point Q is a solution to the martingale problem for $\mathcal{A}_\ell$ where $(\mathcal{A}_\ell f)(x) = \phi_\ell(x)(\mathcal{A}f)(x)$. In particular any such Q must agree with P_x on $\mathcal{F}_{\tau_\ell}$. Since the set $\{\omega : \sup_{0\le t\le T} |x(t)| \ge \ell$ is in $\mathcal{F}_{\tau_\ell}$ and is closed in Ω,

$$\limsup_{h\to 0} P_{h,x}^\ell \left[\sup_{0\le t\le T} |x(t)| \ge \ell \right] \le P_x \left[\sup_{0\le t\le T} |x(t)| \ge \ell \right]$$

Since P_x is a diffusion with bounded coefficients

$$\limsup_{\ell\to\infty} P_x \left[\sup_{0\le t\le T} |x(t)| \ge \ell \right] = 0$$

Therefore the difference between $P^{\ell}_{h,x}$ and $P_{h,x}$ on $\mathcal{F}_T$ goes to 0 as $h \to 0$. This proves the convergence of $\{P_{h,x}\}$ to $\{P_x\}$ as $h \to 0$.

Step 3. We now show that, for fixed ℓ and x, the family $\{P^{\ell}_{h,x} : h > 0\}$ is totally bounded. The basic estimate is on the following quantity:

$$\psi(t,\delta) = \sup_{h>0} \sup_{x \in R^d} P^{\ell}_{h,x}\left[\sup_{1 \le j \le \frac{t}{h}} |x(jh) - x(0)| \ge \delta\right]$$

So long as t is a mutiple of h, this is the same as

$$\psi(t,\delta) = \sup_{h>0} \sup_{x \in R^d} P^{\ell}_{h,x}\left[\sup_{1 \le s \le t} |x(s) - x(0)| \ge \delta\right]$$

Note that if $|x| \ge 2\ell$ then

$$P^{\ell}_{h,x}\left[\sup_{1 \le j \le \frac{t}{h}} |x(jh) - x(0)| \ge \delta\right] = 0$$

Let f be a smooth function which is 1 on a ball of radius δ around some point x_0 and 0 outside a ball of radius 2δ. Denote by

$$C_f = \sup_{h>0} \sup_{x} \frac{1}{h} \int [f(y) - f(x)] \pi^{\ell}_h(x, dy)$$

which is finite because of our assumption. Then with respect to any $P^{\ell}_{h,x}$,

$$A_f(nh) = f(x(nh)) - f(x(0)) + nhC_f$$

is a submartingale. Clearly $A(0) = 0$, and if $|x - x_0| \le \delta$ for the stopping time $\tau = \inf\{j : |x(jh)| \ge 2\delta\}$,

$$\begin{aligned}
P^{\ell}_{h,x}[\tau \le nh] &\le P^{\ell}_{h,x}[f(x(\tau \wedge nh)) = 0] \\
&\le P^{\ell}_{h,x}[f(x(0)) - f(x(\tau \wedge nh)) = 1] \\
&\le E^{P^{\ell}_{h,x}}[f(x(0)) - f(x(\tau \wedge nh))] \\
&\le E^{P^{\ell}_{h,x}}[C_f \tau \wedge nh - A(\tau \wedge nh)] \\
&\le E^{P^{\ell}_{h,x}}[C_f \tau \wedge nh] \\
&\le C_f nh
\end{aligned}$$

Since the ball $|x| \le 2\ell$ can be covered by a finite number of such balls, we need only a finite number of functions f. There is therefore a finite constant $C_{\delta,\ell}$ such that $\psi(t,\delta) \le C_{\delta,\ell} t$.

In order to prove the total boundedness we need to estimate the modulus of continuity $\Delta(\omega)$ of the path $\omega = x(\cdot)$ in $[0,T]$.

$$\Delta(\omega,\delta) = \sup_{\substack{0 \le s \le t \le T \\ |s-t| \le \delta}} |x(t) - x(s)|$$

Let us define $k_1 = \inf\{j : |x(jh) - x(0)| \geq \delta\}$, $k_2 = \inf\{j : |x((k_1 + j)h) - x(k_1 h)| \geq \delta\}$ and so on. Let us consider for an integer N, the sum $k_N = k_1 + k_2 + \cdots + k_N$ and $m_N = \min(k_1, k_2, \ldots, k_N)$. Suppose $0 \leq j_1 \leq j_2 \leq k_N$, $j_2 - j_1 \leq m_N$ and $k_N \geq k$ where $kh = T$. There can be atmost one partial sum $k_1 + k_2 + \cdots + k_r$ between j_1 and j_2. Moreover if we denote by $\eta = \sup_{0 \leq j \leq k-1} |x(jh) - x((j+1)h)|$, then

$$|x(j_1 h) - x(j_2 h)| \leq 4\delta + \eta$$

and hence

$$\Delta(\omega, hm_N) \leq 4\delta + \eta$$

We are almost done. We have uniform conditional estimates on $k_1, k_2, \ldots$ of the type

$$P[hk_{i+1} \leq t | k_1, \ldots, k_i] \leq Ct$$

which implies that

$$E[e^{-hk_{i+1}} | k_1, k_2 \ldots, k_i] \leq \rho < 1$$

Therefore

$$E[e^{-h(k_1 + k_2 + \cdots + k_N)}] \leq \rho^N$$

and

$$P[h(k_1 + k_2 + \cdots + k_N) \leq T] \leq e^T \rho^N$$

On the other hand

$$P[hm_N \leq \epsilon] \leq NC\epsilon$$

Finally,

$$P[\Delta(\omega, \epsilon) \geq 5\delta] \leq P[\eta \geq \delta] + NC\epsilon + e^T \rho^N$$

We can pick N and then ϵ to control it provided we control

$$P[\eta \geq \delta] \leq [\frac{T}{h}] \ \pi^{\ell}_{h,x}(x, B(x, \delta)^c)$$

The locality of the operator $\mathcal{A}$ gaurantees that the limit

$$\begin{aligned}
\lim_{h \to 0} \sup_{|x - x_0| \leq \delta} & \frac{1}{h} \pi^{\ell}_{h,x}(x, B(x, 3\delta)^c) \\
& \leq \lim_{h \to 0} \sup_{|x - x_0| \leq \delta} \frac{1}{h} \pi^{\ell}_{h,x}(x, B(x_0, 2\delta)^c) \\
& \leq \lim_{h \to 0} \sup_{|x - x_0| \leq \delta} \frac{1}{h} \int [f(y) - f(x)] \pi^{\ell}_{h,x}(x, dy) \\
& = 0
\end{aligned}$$

□

4.9 Explosion.

Just as a solution to the ODE $\dot{x} = b(x)$ can explode at a finite time, diffusion processes with unbounded coffecients can explode as well at a finite random time. In order to define and study explosion, we need the notion of a local solution, since global solutions by definition are defined for all $t \geq 0$ and cannot explode. We have the natural stopping time $\tau_\ell = \inf\{t : |x(t)| \geq \ell\}$ and the corresponding σ-field $\mathcal{F}_{\tau_\ell}$. A local solution for $\mathcal{A}$ is a family of measures P_ℓ on $\mathcal{F}_{\tau_\ell}$ that are consistent, i.e. $P_{\ell+1} = P_\ell$ on $\mathcal{F}_{\tau_\ell}$. We can abuse notation and denote all of them by P. Although P is well defined on the field $\widehat{\mathcal{F}} = \cup_\ell \mathcal{F}_{\tau_\ell}$ it may not be countably additive on $\widehat{\mathcal{F}}$. It is not hard to check that in order that P be countably additive on $\widehat{\mathcal{F}}$ it is necessary and sufficient that

$$\lim_{\ell\to\infty} P[\tau_\ell \leq T] = 0$$

for every $T < \infty$. This is seen to be equivalent to $\tau_\ell \to \infty$ in probability as $\ell \to \infty$. The quantity

$$\lim_{\ell\to\infty} P[\tau_\ell \leq t] = F(t)$$

defines the distribution function of the 'explosion' time, and we need to show that $F(t) \equiv 0$ to avoid explosion in a finite time with positive probability. If the explosion has probability 0, then P extends uniquely as a countably additive measure on $(\Omega, \mathcal{F})$. After all the field $\widehat{\mathcal{F}}$ generates the σ-field $\mathcal{F}$. We need conditions for nonexplosions.

Theorem 4.10. *Suppose there exists a smooth function $u(x)$ such that $u(x) \geq 0$, $u(x) \to \infty$ as $|x| \to \infty$ and $(\mathcal{A}u)(x) \leq Cu(x)$ for some $C < \infty$. Then any local solution for $\mathcal{A}$ cannot explode.*

Proof. We consider

$$Z_t = e^{-Ct} u(x(t))$$

By Itô's formula or the martingale formulation, Z_t is a supermartingale upto any stopping time $\tau_\ell = inf\{t : |x(t)| \geq \ell\}$. In particular $E[Z_{\tau_\ell}] \leq E[Z_0] = u(x)$. On the other hand

$$Z_{\tau_\ell} = e^{-C\tau_\ell} u(x(\tau_\ell)) \geq e^{-C\tau_\ell} \inf_{|y|\geq\ell} u(y) = e^{-C\tau_\ell} c_\ell$$

where $c_\ell \to \infty$ as $\ell \to \infty$. This means

$$E[e^{-C\tau_\ell}] \leq \frac{u(x)}{c_\ell}$$

From the simple bound

$$P[\tau_\ell \leq T] \leq e^{CT} E[e^{-C\tau_\ell}]$$

it folows that there cannot be an explosion. □

Corollary 4.11. *If* $\|a(x)\| \leq C(1+|x|^2)$ *and* $\|b(x)\| \leq C(1+|x|)$ *there cannot be an explosion.*

Proof. Let us try $U(x) = (1+|x|^2)$. Each derivative lowers the power by 1 and therfore $\mathcal{A}U$ again has atmot quadratic growth. We are done. □

Remark 4.16. We can deal with time dependent coefficients with no additional work. We can apply the same method or think of time as an extra space coordinate (with index 0), with the corresponding $a_{0,j} \equiv 0$ and $b_0 = 1$.

Remark 4.17. In Theorem 4.9 it is enough to assume that the process corresponding to the limiting $\mathcal{A}$ does not explode. Bounded is really not needed.

Remark 4.18. Uniqueness is a local issue. If in some neighborhood of each point the given coefficients are the restrictions of other coefficients for which uniqueness holds, then uniqueness is valid for the given set of coefficients.

Exercise 4.2. We can provide conditions for explosion. If $U(x) > 0$ and is bounded on R^n and satisfies $(\mathcal{A}U)(x) \geq cU(x)$ for some $c > 0$, then the process explodes with positive probability. Use the reverse inequalities in the proof of nonexplosion to get a uniform lower bound on $E[e^{-c\tau_\ell}]$.

Exercise 4.3. Show that the process in 1 dimension corresponding to $\frac{1}{2}\frac{d^2}{dx^2} + x^2\frac{d}{dx}$ explodes.

Exercise 4.4. Does the process corresponding to $\frac{e^{x^2}}{2}\Delta$ explode in dimension $d = 2$? How about $d \geq 3$? (Hint: use random time change). Can any process corresponding to $[a, 0]$ with continuous positive a explode in $d = 1$ or 2?