Assignment 4. Due Oct 7, 2003.

Construct the Lebesgue measure (area) on the Borel subsets of the square $\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ along the foloowing steps.

1. Start with Borel rectangles, i.e. sets of the form $\{E = (x, y) : x \in A, y \in B\}$ where A and B are chosen from $\mathcal{B}_1$ the class of Borel subsets of $[0, 1]$. Show that the class of sets $\mathcal{F}$ that are finite disjoint unions of Borel rectangles is a field and define the Borel σ-field $\mathcal{B}_2$ in the square as the smallest σ-field generated by this field.

2. Define for any set $E \in \mathcal{B}_2$ the sections

$$E_x = \{y : (x, y) \in E\} \qquad E_y = \{x : (x, y) \in E\}$$

Show that E_x, E_y are in $\mathcal{B}_1$ and $m(E_x)$, $m(E_y)$ are measurable functions of x and y. More over

$$\int m(E_x)dx = \int m(E_y)dy$$

Show that their common value $m_2(E)$ defines a countably additive meausure, i.e. the Lebesgue measure on the Borel subsets of $[0, 1] \times [0, 1]$. [Hint: in verifying properties for arbitrary $E \in \mathcal{B}_2$, check that the class of stes for which the property holds is a monotone class that contains $\mathcal{F}$.]