Countable product measures.

Let $X = \prod X_i$ be a product space. $X \ni x = \{x_j\}$ where $x_j \in X_j$. Σ_j is a σ-field of subsets of X_j and μ_j is a countably additive measure on (X_j, Σ_j) with $\mu(X_j) = 1$ for every j. A finite dimensional cylinder set is a set of the form $A = B \times X_{n+1} \times X_{n+2} \times \cdots$ where $B \in \Sigma_1 \times \cdots \times \Sigma_n$. Such sets A form a field $\mathcal{F}$ of subsets of X and the σ- field Σ generated by them is called the product σ-filed $\prod \Sigma_i$. We want to construct a measure μ on (X, Σ) which will be $\prod \mu_i$. We define $\mu(A) = (\mu_1 \times \mu_2 \cdots \mu_n)(B)$ if $A \in \mathcal{F}$ is of the form $A = B \times X_{n+1} \times X_{n+2} \times \cdots$. We can use a larger n and $A = B_m \times X_{m+1} \times X_{m+2} \times \cdots$ where $B_m = B \times X_{n+1} \times X_{n+2} \times \cdots \times X_m$. The definition is consistent, because $\mu_j(X_j) = 1$. This is used to prove the finite additivity of μ on $\mathcal{F}$.

The crucial step is to prove the countable additivity of μ on $\mathcal{F}$. Let $A_n \in \mathcal{F}$, $A_n \downarrow$, $\mu(A_n) \geq \epsilon$ for some $\epsilon > 0$ and for all n. Then $\cap_n A_n \neq \emptyset$. With out loss of generality assume $A_n = B_n \times X_{n+1} \times X_{n+2} \times \cdots$ for some $B_n \in \prod_1^n \Sigma_i$. We denote by $B_{n,x_1} = \{(x_2, \ldots, x_n) : (x_1, x_2, \ldots, x_n) \in B_n\}$.

$$\mu(A_n) = \int (\mu_2 \times \cdots \times \mu_n)(B_{n,x_1}) d\mu(x_1) \geq \epsilon$$

It follows that

$$\mu_1\{x_1 : (\mu_2 \times \cdots \times \mu_n)(B_{n,x_1}) \geq \frac{\epsilon}{2}\} \geq \frac{\epsilon}{2}$$

Since A_n is $\downarrow$ it follows that $B_{n+1} \subset B_n \times X_{n+1}$ and therefore

$$(\mu_2 \times \cdots \times \mu_n)(B_{n,x_1}) \geq (\mu_2 \cdots \mu_{n+1})(B_{n+1,x_1})$$

The function

$$(\mu_2 \cdots \mu_n)(B_{n,x_1})$$

is non-increasing with n and therefore

$$\{x_1 : (\mu_2 \times \cdots \times \mu_n)(B_{n,x_1}) \geq \frac{\epsilon}{2}\}$$

is $\downarrow$. Hence there is an $\bar{x}_1$ such that for all $n \geq 2$

$$(\mu_2 \times \cdots \times \mu_n)(B_{n,\bar{x}_1}) \geq \frac{\epsilon}{2}$$

Repeating the argument there exist $\bar{x}_2$ such that for $n \geq 3$,

$$(\mu_3 \times \cdots \times \mu_n)(B_{n,\bar{x}_1,\bar{x}_2}) \geq \frac{\epsilon}{2^2}$$

By induction, having chosen $(\bar{x}_1, \ldots, \bar{x}_{k-1})$ there exists $\bar{x}_k$ such that for $n \geq k+1$

$$(\mu_{k+1} \times \cdots \times \mu_n)(B_{n,\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_k}) \geq \frac{\epsilon}{2^k}$$

The sequence $x = \{\bar{x}_k\}$ has the property: for $n \geq k+1$

$$B_{n,\bar{x}_1,\ldots,\bar{x}_k} \neq \emptyset$$

In particular

$$B_{k+1,\bar{x}_1,\ldots,\bar{x}_k} \neq \emptyset$$

Since $B_{k+1} \subset B_k \times X_{k+1}$, we have for all k

$$(\bar{x}_1, \ldots, \bar{x}_k) \in B_k$$

Since $B_k \in \prod_1^k \Sigma_i$, this implies

$$x = \{\bar{x}_i\} \in A_k \quad \text{for all } k$$

Hence $\cap_k A_k \neq \emptyset$.