One of the goals of the course is to learn how to present proofs in a logically precise and easily understandable form. Answers submitted in sloppy form will be returned even if correct!

Example: Page 39 problem 6.

$$\limsup x_n + \liminf y_n \le \limsup(x_n + y_n) \le \limsup x_n + \limsup y_n$$

Proof: Part1. Let $a = \limsup x_n$ and $b = \limsup y_n$. By definition

$$a = \inf_n \sup_{m \ge n} x_m$$

Since $\sup_{m\ge n} x_m$ is non-increasing in n, Given any $\epsilon > 0$, there exists $n_1(\epsilon)$ such that for $n \ge n_1$

$$x_n \le \sup_{m\ge n} x_m \le a + \frac{\epsilon}{2}$$

Similarly, there exists $n_2(\epsilon)$ such that for $n \ge n_2$

$$y_n \le \sup_{m\ge n} y_m \le b + \frac{\epsilon}{2}$$

If we define $n_0(\epsilon) = \max\{n_1(\epsilon), n_2(\epsilon)\}$, for $n \ge n_0$ we have

$$x_n + y_n \le a + b + \epsilon$$

which implies

$$\limsup(x_n + y_n) \le a + b + \epsilon$$

Since $\epsilon > 0$ is arbitrary, this implies

$$\limsup(x_n + y_n) \le a + b = \limsup x_n + \limsup y_n$$

Part 2. First we show that if $a = \limsup x_n$, then for any $\epsilon > 0$, there exists an infinite number of j such that $x_j \ge a - \frac{\epsilon}{2}$. If not there would be a finite N such that for $n > N$, $x_n \le a - \frac{\epsilon}{2}$ and this would imply that $\sup_{m\ge n} x_m \le a - \frac{\epsilon}{2}$ for all $n > N$ and hence $\limsup x_n \le a - \frac{\epsilon}{2}$, a contradiction. If $c = \liminf y_n$, for any $\epsilon > 0$ there exists $n_3(\epsilon)$ such that for $n \ge n_3$

$$y_n \ge \inf_{m\ge n} y_m \ge c - \frac{\epsilon}{2}$$

With an infinite number of n for which $x_n > a - \frac{\epsilon}{2}$ and $y_n \ge c - \frac{\epsilon}{2}$ for sufficiently large n it follows that $x_n + y_n \ge a + c - \epsilon$ for infinitely many n. This in turn implies that

$$\limsup(x_n + y_n) \ge a + c - \epsilon$$

Since $\epsilon > 0$ is arbitrary it follows that

$$\limsup(x_n + y_n) \ge a + c = \limsup x_n + \liminf y_n$$

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