Real Variables Fall 2007.

Assignment 6. Due Oct 15.

Problem 1. We have a measure space $(X, \mathcal{F}, \mu)$ which is not a finite set of points. More precisely assume that there is a countable collection of disjoint measurable sets $\{A_j\}$ such that $0 < \mu(A_j) < \infty$ for each j. The goal is to prove that in such a case not every bounded linear functional $\Lambda(f)$ defined on $L_\infty(X, \mathcal{F}, \mu)$ is of the form

$$\Lambda(f) = \int_X f(x)\, \phi(x) d\mu$$

for some $\phi(x) \in L_1(X, \mathcal{F}, \mu)$. This is to be carried out in several steps.

Step 1. Find a closed subspace $B \subset L_\infty(X, \mathcal{F}, \mu)$ which is isomorphic to ℓ_∞, the space of all bounded sequences $\{a_n\}$ of real numbers with $\|\{a_n\}\| = \sup_{1 \le n < \infty} |a_n|$.

Step 2. Use Hahn-Banach theorem to construct a linear functional on ℓ_∞ satisfying

$$\liminf a_n \le \Lambda(\{a_n\}) \le \limsup a_n$$

Step 3. Show that $\Lambda(\{a_n\}))$ cannot be of the form

$$\Lambda(\{a_n\})) = \sum_n a_n p_n$$

for some sequence $\{p_n\}$ with $\sum_n |p_n| < \infty$.

Step 4. Transplant Λ on B and extend to $L_\infty(X, \mathcal{F}, \mu)$ again by Hahn-Banach theorem. Show that the extension cannot be of the form

$$\Lambda(f) = \int_X f(x)\, \phi(x) d\mu$$

for some $\phi(x) \in L_1(X, \mathcal{F}, \mu)$.

Problem 2. Let $(X, \mathcal{F}.\mu)$ be a non-atomic measure space (i.e. one in which every set A of finite non-zero measure can be divided into two disjoint sets whose measures are equal) which is σ-finite but NOT finite, i.e. $\mu(X) = \infty$. If $1 \le p_1 < p_2 < \infty$, show that there are functions f_1, f_2 such that

$$f_1 \in L_{p_1}(X, \mathcal{F}.\mu), \notin L_{p_2}(X, \mathcal{F}.\mu)$$

and

$$f_2 \in L_{p_2}(X, \mathcal{F}.\mu), \notin L_{p_1}(X, \mathcal{F}.\mu)$$

What is the situation when $\mu(X) < \infty$?