L_p spaces are uniformly convex.

The case $p \geq 2$.

Lemma: Given $p \geq 2$, there exists a constant $c = c(p) > 0$ such that for all real numbers f, g

$$\left|\frac{f+g}{2}\right|^p + c\left|\frac{f-g}{2}\right|^p \leq \frac{|f|^p}{2} + \frac{|g|^p}{2}$$

Proof: It is valid with $c = 1$ if $f = 0$. We can assume with out loss of generality that $f > 0$. Dividing through by f and denoting $\frac{f}{g}$ by x, we need

$$\left|\frac{1+x}{2}\right|^p + c\left|\frac{1-x}{2}\right|^p \leq \frac{1}{2} + \frac{|x|^p}{2}$$

or

$$F(x) = \frac{1}{2} + \frac{|x|^p}{2} - \left|\frac{1+x}{2}\right|^p \geq c\left|\frac{1-x}{2}\right|^p = G(x)$$

for $-\infty < x < \infty$. Clearly $F(x) > 0$ unless $x = 1$ when $F(1) = 0$. Check that $F''(1) = \frac{p(p-1)}{4} > 0$. $G(x)$ which is also 0 only at $x = 1$ vanishes faster there than $F(x)$. Near $\pm\infty$, they are both asymptotic to multiples $|x|^p$ and hence $\frac{G(x)}{F(x)}$ is bounded above by some $\frac{1}{c}$. Now

$$\left|\frac{f(x)+g(x)}{2}\right|^p + c\left|\frac{f(x)-g(x)}{2}\right|^p \leq \frac{|f(x)|^p}{2} + \frac{|g(x)|^p}{2}$$

Integrating

$$\left\|\frac{f+g}{2}\right\|_p^p + c\left\|\frac{f-g}{2}\right\|_p^p \leq \frac{\|f\|_p^p}{2} + \frac{\|g\|_p^p}{2}$$

In particular if $\|f\|_p = \|g\|_p = 1$ and $\left\|\frac{f+g}{2}\right\|_p^p \geq 1 - \delta$ then $\left\|\frac{f-g}{2}\right\|_p^p \leq c^{-1}\delta$. This proves uniform convexity.

The case $p < 2$. If we define $G(x) = \frac{|1-x|^2}{(1+|x|)^{2-p}}$, then we can show $F(x) \geq c\,G(x)$. The extra $(1+|x|)^{2-p}$ in the denominator adjusts the behavior at ∞. We start with

$$\int \frac{|f(x)-g(x)|^2}{|f(x)|^{2-p} + |g(x)|^{2-p}} d\mu \leq c^{-1}\delta$$

We estimate

$$\begin{aligned}
\int |f(x)-g(x)|^p d\mu &= \int \frac{|f(x)-g(x)|^p}{[|f(x)|^{2-p} + |g(x)|^{2-p}]^{\frac{p}{2}}} [|f(x)|^{2-p} + |g(x)|^{2-p}]^{\frac{p}{2}} d\mu \\
&\leq \left[\int \left[\frac{|f(x)-g(x)|^p}{[|f(x)|^{2-p} + |g(x)|^{2-p}]^{\frac{p}{2}}}\right]^{\frac{2}{p}} d\mu\right]^{\frac{p}{2}} \\
&\times \left[\int \left[[|f(x)|^{2-p} + |g(x)|^{2-p}]^{\frac{p}{2}}\right]^{\frac{2}{2-p}} d\mu\right]^{1-\frac{p}{2}} \\
&\leq (c^{-1}\delta)^{\frac{p}{2}} \times C\left[\int [|f(x)|^p + |g(x)|^p] d\mu\right]^{1-\frac{p}{2}}
\end{aligned}$$

Done!