Real Numbers. Properties.

1. Field. Addition, 0, additive inverse (Abelian Group) Multiplication, 1, leaving 0 out (Abelian Group) Algera. Distributive $a(b+c) = ab + ac$. Add, subtract, multiply, and divide by anything other than 0.

2. Ordered Field.

$$\mathbf{R}\backslash\{0\} = \mathbf{R}^+ \cup \mathbf{R}^-$$

$$a, b \in \mathbf{R}^+ \Rightarrow a + b, ab \in \mathbf{R}^+$$

We say $a > b$ if $a - b \in \mathbf{R}^+$
Upper Bounds, Lower bounds,
Rationals $\mathbf{Q}$ satisfy. But LUB, GLB exist only for $\mathbf{R}$.

3. Consequences.
Every bounded sequence has a convergent subsequence
Every bounded monotone sequence converges.
Any open covering of a bounded closed set has a finite sub cover.

We begin with integration. Riemann Integrals. Lebesgue integrals.

The notion of "length" of a set. Let us stick to the interval $[0, 1]$ we try to define $\mu(A)$ which we think of the length of the set A. If A is an interval $[a, b] \subset [0, 1]$ the $\mu(A) = b - a$.

Let us define for any set $A \subset [0, 1]$, $\mu^*(A)$ by

$$\mu^*(A) = \inf \big[\sum_{j=1}^{\infty} \mu(I_j) : \cup_j I_j \supset A \big]$$

μ^* is finitely as well as countably subadditive.

Properties of μ^*.

1. $\mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B)$

2. $\mu^*(\cup_j A_j) \leq \sum_j \mu^*(A_j)$

3. $\mu^*([a, b]) = (b - a)$

The first two properties are easily proved. Cover A, B by intervals and the combined set of intervals covers $A \cup B$. Split ϵ into $\frac{\epsilon}{2}$ margin for each. For the countable case make the margin $\frac{\epsilon}{2^j}$ for A_j. More precisely given $\epsilon > 0$ there are intervals $\{I_{j,k} = (a_{j,k}, b_{j,k})\}$ such that for each j

$$\cup_k I_{j,k} \supset A_j$$

and

$$\sum_k (b_{j,k} - a_{j,k}) \leq \mu^*(A_j) + \epsilon 2^{-j}$$

$$\cup_{j,k} I_{j,k} \supset \cup_j A_j$$

and

$$\mu^*(\cup_j A_j) \le \sum_{j,k}(b_{j,k} - a_{j,k}) \le \sum_j [\mu^*(A_j) + \epsilon 2^{-j}] = \sum_j \mu^*(A_j) + \epsilon$$

Finally if $\{(a_j, b_j)\}$ covers $[a, b]$ there is a finite sub cover.

$$[a, b] \subset \cup_{i=1}^n (a_i, b_i)$$

with out loss of generality we can assume

$$a_1 < a < a_2 < b_1 < a_3 < \cdots < a_n < b_{n-1} < b < b_n$$

and

$$(b - a) \le \sum_i (b_i - a_i)$$

$\mu^*([a, b]) \ge (b - a)$. But $\mu^*([a, b]) \le b - a$.

Let us define the class Σ of subsets of $[0, 1]$. $E \in \Sigma$ if for every subset $A \subset [0, 1]$

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) = \mu^*(A)$$

Since

$$\mu^*(A) \le \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

we only need to prove

$$\mu^*(A) \ge \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Properties of μ^* on Σ

1. Intervals $[a, b] \in \Sigma$

2. Σ is closed under finite as well as countable unions and complementation and therefore under countable intersections ($\sigma-$field.)

3. $\mu^*(E)$ on Σ is countably additive, i.e., if $\{Ej\}$ are disjoint, then

$$\mu^*(\cup_j E_j) = \sum_j \mu^*(E_j)$$

The notion of length is well defined on Σ, extending it from intervals.

Proof.

1. Let A be arbitrary. Let $\mu^*(A) = m$. Then given $\epsilon > 0$, there are intervals $\{I_j\} = \{(a_j, bj)\}$ such that $A \subset \cup_j (a_j, b_j)$ and

$$\sum_j (b_j - a_j) \le m + \epsilon$$

If $E = [a,b]$, then $E^c \subset [0,a] \cup [b,1] = E_1 \cup E_2$. Each I_j is the union of three essentially disjoint intervals $I_j \cap E$, $I_j \cap E_1$ and $I_j \cap E_2$. If $\{I_j\}$ covers A then $\{I_j \cap E\}$ covers $A \cap E$ and $\{I_j \cap E_1\}$ and $\{I_j \cap E_2\}$ together cover $A \cap E^c$. It is now clear that for any A,

$$\begin{aligned} m + \epsilon &\geq \sum_{j=1}^{\infty} [\mu^*(I_j \cap E_1) + \mu^*(I_j \cap E_2) + \mu^*(I_j \cap E_3)] \\ &\geq \mu^*(A \cap [a,b]) + \mu^*(A \cap [a,b]^c) \end{aligned}$$

i.e. $[a,b] \in \Sigma$.

2. Since the definition is symmetric in E and E^c it follows that if $E \in \Sigma$ so does E^c. Assume

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) = \mu^*(A)$$

for all A. Replace A by $A \cap F$ and by $A \cap F^c$, to get

$$\mu^*(A \cap E \cap F) + \mu^*(A \cap E^c \cap F) = \mu^*(A \cap F)$$

and

$$\mu^*(A \cap E \cap F^c) + \mu^*(A \cap E^c \cap F^c) = \mu^*(A \cap F^c)$$

for all A. Adding them

$$\mu^*(A \cap E \cap F) + \mu^*(A \cap E^c \cap F) + \mu^*(A \cap E \cap F^c) + \mu^*(A \cap E^c \cap F^c) = \mu^*(A)$$

We note that $(E \cap F) \cup (E^c \cap F) \cup (E \cap F^c) = E \cup F$ and $(E \cup F)^c = E^c \cap F^c$. Using sub additivity

$$\mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^c) \leq \mu^*(A)$$

which is the hard part. So $E \cup F \in \Sigma$. So does $E \cap F$. If E and F are disjoint, taking $A = A \cap (E \cup F)$

$$\mu^*(A \cap E) + \mu^*(A \cap F) + \mu^*(A \cap (E \cup F)^c) = \mu^*(A)$$

Taking $A = [0,1]$,

$$\mu^*(E) + \mu^*(F) = \mu^*(E \cup F)$$

Finally we want to prove that if E_j is a countable collection, mutually disjoint, and $E_j \in \mathcal{E}$ for every j, then $\cup_{j=1}^{\infty} E_j = E \in \Sigma$ and

$$\mu^*(E) = \sum_{j=1}^{\infty} \mu^*(E_j)$$

Let $F_n = \cup_{j=1}^n E_j$, then

$$\sum_{j=1}^n \mu^*(A \cap E_j) + \mu^*(A \cap F_n^c) = \mu^*(A)$$

and since $F_n^c \supset E^c$,

$$\sum_{j=1}^\infty \mu^*(A \cap E_j) + \mu^*(A \cap F^c) \leq \mu^*(A)$$

By subadditivity

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) \leq \mu^*(A)$$

Proves $E \in \Sigma$ and also

$$\sum_{j=1}^n \mu^*(E_j) \leq \mu^*(E)$$

and letting $n \to \infty$

$$\sum_{j=1}^\infty \mu^*(E_j) \leq \mu^*(E)$$

But the other side is true by sub-addivity. So μ^* is defined on Σ as a countably additive measure agreeing with length on $[0,1]$. This is Lebesgue measure. We will call it μ.

A class of sets is a $\sigma-$field if it is closed under countable unions and complementation. Given a collection $\mathcal{A}$ there is a smallest $\sigma-$field containing $\mathcal{A}$, called the $\sigma-$field generated by $\mathcal{A}$ denoted by $\sigma(\mathcal{A})$. If we denote by $\mathcal{I}$ the collection of intervals the the Borel $\sigma-$field $\mathcal{B}$ is $\sigma(\mathcal{I})$. The Lebesgue measure is defined on $\Sigma \supset \mathcal{B}$.

Fact. A monotone class is closed under increasing and decreasing limits. A monotone field is a $\sigma-$field and a $\sigma-$field is a monotone class. The smallest monotone class containing a field is the same as the $\sigma-$field generated by it. In particular if two measures agree on a field they agree on the $\sigma-$field generated by the field. Lebesgue measure is unique on $\mathcal{B}$.

Let $\mathcal{F}$ be a field and $\mathcal{M}(\mathcal{F})$ be the monotone class generated by $\mathcal{F}$. Then $\mathcal{M} = \sigma(\mathcal{F})$. To see this let us define for sets E,

$$\mathcal{M}(E) = \{F : E \cap F^c, F \cap E^c, E \cup F \in \mathcal{M}\}$$

$\mathcal{M}(E)$ is a monotone class, and for $E \in \mathcal{F}$, contains $\mathcal{F}$ and so contains $\mathcal{M}$. In other words if $E \in \mathcal{F}$ and $F \in \mathcal{M}$ then $E \cap F^c, F \cap E^c, E \cup F \in \mathcal{M}$. The relation is symmetric. Therefore if $F \in \mathcal{F}$ and $E \in \mathcal{M}$ then $E \cap F^c, F \cap E^c, E \cup F \in \mathcal{M}$. In other words $\mathcal{M}(E) \supset \mathcal{F}$ for $E \in \mathcal{M}$. Hence $\mathcal{M}(E) \supset \mathcal{M}$. Finally $E, F \in \mathcal{M}$ implies $E \cap F^c, F \cap E^c, E \cup F \in \mathcal{M}$ or $\mathcal{M}$ is a field. $\mathcal{M}(\mathcal{F}) \supset \sigma(\mathcal{F})$ and $\mathcal{M}(\mathcal{F}) \subset \sigma(\mathcal{F})$.

Construction of measures.

A set function μ defined for $A \in \mathcal{F}$, a field of subsets of a space $\mathbf{X}$, satisfying

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

for sets A_i that are pairwise disjoint and $\{A_i\}, A \in \mathcal{F}$, is called a countably additive measure on $\mathcal{F}$. A countably additive measure μ on $\mathcal{F}$ extends uniquely as a countably additive measure to $\sigma(\mathcal{F})$, the $\sigma-$field generated by $\mathcal{F}$.

Repeat the proof for Lebesgue measure with slight changes.

A semiring $\mathcal{S}$ of subsets of a set $\mathbf{X}$ satisfies, $A, B \in \mathcal{S}$ implies $A \cap B \in \mathcal{S}$. $\mathbf{X} \in \mathcal{S}$. $A, B \in \mathcal{S}$, $A \subset B$ implies $B - A = A_1 \cup \cdots \cup A_k$ where $\{A_i\}$ are disjoint and $A_i \in \mathcal{S}$ for $1 \leq i \leq k$.

A set function μ defined for $A \in \mathcal{S}$, a semiring of subsets of a space $\mathbf{X}$, satisfying

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

for sets A_i that are pairwise disjoint and $\{A_i\}, A \in \mathcal{S}$, is called a countably additive measure on $\mathcal{S}$. A countably additive measure μ on $\mathcal{S}$ extends uniquely as a countably additive measure to $\sigma(\mathcal{S})$, the $\sigma-$field generated by $\mathcal{S}$.

Disjoint union of sets from $\mathcal{S}$ is a field $\mathcal{F}(\mathcal{S})$ and μ extends naturally as

$$\mu(\cup_{i=1}^{k} A_i) = \sum_{i=1}^{k} \mu(A_i)$$

for disjoint unions. μ is countably additive on $\mathcal{F}(\mathcal{S})$ and extends uniquely to $\sigma(\mathcal{F}(\mathcal{S}))$.

Integration.

What is the class of functions that we can integrate? Given $(\mathbf{X}, \Sigma)$.

A function $f : \mathbf{X} \to R$ is *measurable* if for any $E \in \mathcal{B}(R)$,

$$f^{-1}(E) = \{x : f(x) \in E\} \in \Sigma$$

Enough to check

$$f^{-1}(I) = \{x : f(x) \in I\} \in \Sigma$$

for intervals I of the form $(-\infty, a)$, $a \in R$.

If f and g are measurable then so are $f + g$, fg and $\frac{1}{f}$. For example

$$\{x : f(x) + g(x) < a\} = \cup_{q \in Q}[\{x : f(x) < q\} \cap \{x : g(x) < a - q\}]$$

where Q are the rationals.

More generally a map $f : \mathbf{X} \to \mathbf{Y}$ is measurable relative to $(\mathbf{X}, \Sigma)$ and $(\mathbf{Y}, \mathcal{E})$ if for every $E \in \mathcal{E}$

$$f^{-1}(E) = \{x : f(x) \in E\} \in \mathcal{E}$$

If $f : \mathbf{X} \to \mathbf{Y}$ and $g : \mathbf{Y} \to \mathbf{Z}$ are measurable relative to $(\mathbf{X}, \Sigma)$, $(\mathbf{Y}, \mathcal{E})$ and $(\mathbf{Z}, \mathcal{F})$, then so is $g \circ f : \mathbf{X} \to \mathbf{Z}$.

Let $f_n(x)$ be a sequence of measurable maps from $(\mathbf{X}, \Sigma) \to (R, \mathcal{B})$. Then

$$C = \{x : \lim_{n \to \infty} f_n(x) = f(x) \quad \text{exists}\}$$

is a measurable set and $f(x)$ restricted to C is measurable.

$$C = \cap_k \cup_\ell \cap_{\substack{n \geq \ell \\ m \geq \ell}} \{x : |f_n(x) - f_m(x)| \leq \frac{1}{k}\}$$

$$C \cap \{x : f(x) \leq a\} = C \cap \cap_k \cup_\ell \cap_{n \geq \ell} \{x : f_n(x) \leq a + \frac{1}{k}\}$$

A simple function takes values $a_1, \ldots, a_k$ on k disjoint sets $E_1, \ldots, E_k$ that are in Σ and whose union is $\mathbf{X}$. Any linear combination of two simple functions is again a simple function.

Any bounded measurable function is a uniform limit of simple functions. Let f be bounded by M and let $\epsilon > 0$ be given. Divide $[-M, M]$ into $k = [\frac{M}{\epsilon}] + 1$ intervals $\{I_j\}$ of size at most 2ϵ and let $a_1, \ldots, a_k$ be their mid points. Define $f_\epsilon(x) = a_j$ on $\{x : f(x) \in I_j\}$. Then $\sup_x |f_\epsilon(x) - f(x)| \leq \epsilon$ and f is uniformly approximated by f_ϵ.

Clearly the integral of a simple function f equal to a_i on E_i is

$$\sum_{i=1}^{k} a_i \mu(E_i)$$

Any bounded measurable function can be approximated uniformly by simple functions and the integrals have a limit that does not depend on the approximations used. Integral is linear and

$$|\int_{\mathbf{X}} f(x) d\mu| \leq \sup_x |f(x)| \mu(\mathbf{X})$$

$$|\int_{\mathbf{X}} [f(x) - g(x)] d\mu \leq \int_{\mathbf{X}} |f(x) - g(x)| d\mu \to 0$$

if $\sup_x |f(x) - g(x)| \to 0$.

The integral is defined for class of bounded measurable functions $\mathbf{B}(\mathbf{X})$. $L(f) = \int_{\mathbf{X}} f(x) d\mu$ satisfies for $f, g \in \mathbf{B}(\mathbf{X})$ and $a, b \in R$

$$L(af + bg) = aL(f) + bL(g)$$

$$f \geq 0 \Rightarrow L(f) \geq 0$$

$$|L(f)| \leq \mu(\mathbf{X}) \sup_{x \in \mathbf{X}} |f(x)|$$

Can define for $A \in \Sigma$

$$\int_A f(x) d\mu = \int_{\mathbf{X}} \chi_A(x) f(x) d\mu(x)$$

and

$$|\int_A f(x) d\mu| \leq \mu(\mathbf{A}) \sup_{x \in A} |f(x)|$$

We know that if f_n are measurable and $|f_n| \leq M$ and $f_n(x) \to f(x)$ for each x then f is measurable and $|f| \leq M$. $\int f_n d\mu$ and $\int f d\mu$ are all well defined.

Bounded Convergence Theorem.

$$\lim_{n \to \infty} \int_{\mathbf{X}} f_n(x) d\mu = \int_{\mathbf{X}} f(x) d\mu$$

Proof. We saw that

$$\mu[\cap_\ell \cup_{n \geq \ell} \{x : |f_n(x) - f(x)| \geq \frac{1}{k}\}] = 0$$

Therefore by countable additivity

$$\mu[\cup_{n \geq \ell} \{x : |f_n(x) - f(x)| \geq \frac{1}{k}\}] \to 0$$

and

$$\mu[\{x : |f_\ell(x) - f(x)| \geq \frac{1}{k}\}] \to 0$$

as $\ell \to \infty$ for every k.

$$\begin{aligned}
|\int f_\ell(x) d\mu - \int f(x) d\mu| &\leq \int |f_\ell(x) - f(x)| d\mu \\
&= \int_{\{x : |f_\ell(x) - f(x)| \leq \frac{1}{k}\}} |f_\ell(x) - f(x)| d\mu + \int_{\{x : |f_\ell(x) - f(x)| \geq \frac{1}{k}\}} |f_\ell(x) - f(x)| d\mu \\
&\leq \frac{1}{k} \mu(\mathbf{X}) + 2M \mu[\{x : |f_\ell(x) - f(x)| \geq \frac{1}{k}\}]
\end{aligned}$$

Let $\ell \to \infty$ and then $k \to \infty$.