Ergodic Theorems.

(Ω, Σ, μ) is a finite measure space with $\mu(\Omega) = 1$ $T : \Omega \to \Omega$ is a measurable map such that $\int f(\omega) d\mu = \int f(T\omega) d\mu$ or equivalently $\mu(T^{-1}A) = \mu(A)$ for all $A \in \Sigma$. T may or may mot be invertible. The ergodic theorem is a statement of the form

$$\lim_{n\to\infty} (A_n f)(\omega) = \frac{1}{n}[f(\omega) + f(T\omega) + \cdots + f(T^{n-1}\omega)] = g(\omega)$$

exists and identifies $g(\omega)$. There is the question of what is assumed about f and and in what sense the convergence takes place.

Theorem. If $f \in L_p$ for some $p \in [1, \infty)$ then the limit exists in L_p If $\mathcal{I}$ is the σ-field of invariant sets, i.e the σ-field generated by functions g that satisfy $g(T\omega) = g(\omega)$ then g is the Radon-Nikodym derivative $\frac{d\lambda}{d\mu}$ on $\mathcal{I}$ where $\frac{d\lambda}{d\mu} = f$ on Σ. In addition for f in L_1 the limit takes place almost everywhere with respect to μ.

First we consider L_2. The transformation T induces a map $f \to Tf$ in every L_p where $(Tf)(\omega) = f(T\omega)$. It preserves the norm, $\|Tf\|_p = \|f\|_p$. We first consider $p = 2$. T is an isometry. One checks that $\|Tf - f\|_2^2 = \|T^*f - f\|_2^2$. The null space of $T^* - I$ is the same as the null space of $T - I$. The closure of the range of $T - I$ is the orthogonal complement of the null space of $T - I$. So given $\epsilon > 0$ any vector f in L_2 can be written uniquely as $f_1 + f_2$ where $Tf_1 = f_1$ and $f_2 \in \overline{(I - T)L_2}$. Given $\epsilon > 0$, f_2 can be written as $f_3 + f_4$ where $\|f_4\| \le \epsilon$ and $f_3 = (I - T)h$ for some $h \in L_2$. $A_n f_1 = f_1$ for all n. $A_n f_3$ telescopes to $\frac{1}{n}[f_3 - T^n f_3] \to 0$ and $\|A_n f_4\| \le \epsilon$. Follows that A_n converges to the orthogonal projection onto the subspace of invariant functions.

For L_p, first look at L_∞. $\|A_n f\|_\infty \le \|f\|_\infty$ and $A_n f \to g$ in L_2 and therefore in every L_p. Using the fact that L_∞ is dense in L_p and $\|A_n f\|_p \le \|f\|_p$ we prove convergence in every L_p.

Almost everywhere convergence. Needs the following Lemma

Maximal ergodic inequality. For $f \in L_1$, let

$$E_n = \{\omega : \sup_{1\le j\le n} [f(\omega) + f(T\omega) + \cdots + f(T^{j-1}\omega)] \ge 0\}$$

Then

$$\int_{E_n} f(\omega) d\mu \ge 0$$

Proof. We denote by

$$h_n(\omega) = \sup_{1\le j\le n} [f(\omega) + f(T\omega) + \cdots + f(T^{j-1}\omega)]$$

and

$$h_n^+(\omega) = \max\{0, h_n(\omega)\}$$

From the definition

$$h_n(\omega) = f(\omega) + h^+_{n-1}(T\omega)$$

or

$$f(\omega) = h_n(\omega) - h^+_{n-1}(T\omega)$$

On E_n, $h_n(\omega) = h^+_n(\omega)$ and $h^+_{n-1}(\omega) \le h^+_n(\omega)$. Therefore $f(\omega) \ge h_n(\omega) - h^+_n(T\omega)$. Since $E_n = \{\omega : h_n(\omega) \ge 0\}$, it follows that

$$\int_{E_n} h_n(\omega) d\mu = \int_{E_n} h^+_n(\omega) d\mu \ge \int_{T^{-1}E_n} h^+_n(\omega) d\mu = \int_{E_n} h^+_n(T\omega) d\mu$$

Let $f \ge 0$ be from L_1. Then

$$\mu[\omega : \sup_{1\le j\le n} \frac{1}{j}[\sum_{i=1}^{j} f(T^{i-1}\omega)] \ge \ell] \le \frac{\|f\|_1}{\ell}$$

Applyong the maximal inequality with $f - \ell$ as f,

$$\int_{E_n} [f(\omega) - \ell] d\mu \ge 0$$

or

$$\mu(E_n) \le \frac{1}{\ell} \int_{E_n} f d\mu \le \|f\|_1$$

where

$$E_n = \{\omega : \sup_{1\le j\le n} \frac{1}{j}[\sum_{i=1}^{j} f(T^{i-1}\omega)] \ge \ell]\} = \{\omega : \sup_{1\le j\le n} \frac{1}{j}[\sum_{i=1}^{j} (f(T^{i-1}\omega) - \ell)] \ge 0]\}$$

letting $n \to \infty$

$$\mu[\{\omega : \sup_{j\ge 1} \frac{1}{j}[\sum_{i=1}^{j} f(T^{i-1}\omega)] \ge \ell\}] \le \frac{\|f\|_1}{\ell}$$

It is clear that for f of the form $g - Tg + h$ where $Th = h$ and g is bounded we have convergence to h everywhere. Any f in L_1 can be approximated by these. And if $\|f\|_1 < \epsilon$,

$$\mu[\sup_{j\ge 1}[\sum_{i=1}^{j} |f(T^{i-1}\omega)|] \ge \sqrt{\epsilon}\}] \le \sqrt{\epsilon}$$

$$\mu[\limsup (A_n f)(\omega) - \liminf (A_n f)(\omega) \ge 2\sqrt{\epsilon}] \le \sqrt{\epsilon}$$

(Ω, Σ, μ, T) is ergodic if $\mathcal{I}$ is trivial, i.e any f with $Tf = f$ is a constant. Clearly constants are there. Then the limit is the projection on to constants which is seen as $f \to \int f d\mu$.