A signed measure on (X, Σ) is a countably additive set function $\mu(A)$ that can take both positive and negative values. We assume that $\sup_{A\in\Sigma} |\mu(A)| < \infty$.

Example. For $A \in \Sigma$, $\mu(A) = \mu_1(A) - \mu_2(A)$ where μ_1 and μ_2 are non negative measures on Σ.

Theorem. Any signed measure μ can be expressed as the difference of two nonnegative measures μ_1, μ_2 with $\mu_1 \perp \mu_2$ in the sense that there are sets $X_1, X_2 \in \Sigma$ such that $X_1 \cap X_2 = \emptyset$ and $\mu_1(X_1^c) = 0$ and $\mu_2(X_2^c) = 0$

Theorem. An equivalent version of the theorem is that there are two subsets $X_1, X_2 \in \Sigma$ with $X_1 \cap X_2 = \emptyset$ and $X_1 \cup X_2 = X$ such that $\mu(A) \geq 0$ for all $A \in \Sigma, A \subset X_1$ and $\mu(A) \leq 0$ for all $A \in \Sigma, A \subset X_2$.

Proof. A set A is *totally positive* if $\mu(B) \geq 0$ for all $B \subset A, B \in \Sigma$. If $\mu(A) > 0$ but A is not totally positive then there is a set $B_1 \subset A$ with $\mu(B_1) < 0$. Then $\mu(A\backslash B_1) = \mu(A) - \mu(B_1) > \mu(A)$. We can choose B_1 such that $\mu(B_1) \leq \frac{1}{2}\inf_{B\subset A}\mu(B)$. Define $A_1 = A\backslash B_1$ and proceed in a similar fashion. We either arrive at a totally positive set at a finite stage, or we have a decreasing sequence A_k with $\mu(A_k) \geq \mu(A) - \sum_{j=1}^{k}\inf_{B\subset A_{j-1}}\mu(B)$ This forces $\sum_{j=1}^{\infty}\inf_{B\subset A_{j-1}}\mu(B)$ to converge and consequently $\inf_{B\subset A_{j-1}}\mu(B) \to 0$. Therefore If $A^+ = \cap_j A_j$ then $\inf_{B\subset A^+}\mu(B) = 0$ and A^+ is totally positive. We have shown that if $\mu(A) > 0$, it has a subset A^+ that is totally positive with $\mu(A^+) \geq \mu(A)$.

Any finite or countable union of totally positive sets is totally positive. Any subset of the union can be written as a disjoint union of subsets of the individual totally positive sets.

Let us choose the largest totally positive set, i.e. one with the largest measure. Its complement must be totally negative. Proves the second version.

Uniqueness of sets to within sets of measure zero in the second version and uniqueness of measures in the first.

Suppose $\mu_1, \mu_2, \mu_3, \mu_4$ are four measures $\mu_1 \perp \mu_2, \mu_3 \perp \mu_4, \mu_1 - \mu_2 = \mu_3 - \mu_4$ the we must show that $\mu_1 = \mu_3$ and $\mu_2 = \mu_4$. We can re write it as $\mu_1 + \mu_4 = \mu_2 + \mu_3$. μ_1, μ_2 are concentrated on disjoint sets E, E^c and μ_3, μ_4 on disjoint sets F, F^c. On $E \cap F$, μ_2 and μ_4 are 0, so implying $\mu_1 = \mu_3$ on subsets of $E \cap F$. On $E \cap F^c$, μ_2 and μ_3 are 0 making $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$. Similarly on $E^c \cap F$ they are all 0. Finally on $E^c \cap F^c$ we have $\mu_1 = \mu_2 = 0$ and $\mu_3 = \mu_4$. Makes $E = F$ to with in a set of measure 0 under μ_1 and μ_2.

One way of generating new measures from old ones is to define for $A \in \Sigma$

$$\lambda(A) = \int_A f(x)d\mu = \int \chi_A(x)f(x)d\mu$$

where $f(x)$ is integrable with respect to μ. $f = f^+ - f^-$ and $\lambda = \lambda^+ - \lambda^-$ where $f^+(x) = \max\{f(x), 0\}, f^-(x) = -\min\{f(x), 0\}$ and $\lambda^\pm(A) = \int_A f^\pm d\mu$. λ is countably additive, by the dominated convergence theorem.

Given a nonnegative countably additive measure μ and a signed countably additive measure λ on (X, Σ) when can we find a measurable function f such that for all $A \in \Sigma$

$$\lambda(A) = \int_A f(x)d\mu$$

A signed measure λ is said to be *absolutely continuous* with respect to a nonnegative measure μ ($\lambda << \mu$) if for every set with $\mu(A) = 0$, we have $\lambda(A) = 0$.

Radon-Nikodym-Theorem. λ has a representaion $\lambda(A) = \int_A f(x)d\mu$ with an integrable f if and only if it is absolutely continuous with respect to μ.

Proof. If $\mu(A) = 0$ clearly $\int_A f d\mu = 0$ for any integrable f. To prove the converse let us write $\lambda = \lambda^+ - \lambda^-$ supported on disjoint sets X^+ and X^-. If $A \subset X^+$ and $\mu(A) = 0$ then $\lambda(A) = \lambda^+(A) = 0$. So $\lambda^+ << \mu$ Similarly $\lambda^- << \mu$. For proving the theorem we can assume that λ is nonnegative. We will try to find nonnegative functions f such that $\int_A f d\mu \le \lambda(A)$ for all $A \in \Sigma$. Maximize $\int_X f d\mu$ among those. It is easy to see that there is a function f that achieves the maximum. If $\int_A f d\mu = \lambda_1(A)$ then $\lambda_2 = \lambda - \lambda_1 \ge 0$ and $\lambda_2 << \mu$. There is now no nontrivial f with $\int_A f d\mu \le \lambda_2(A)$. We can try with $(\lambda_2 - \epsilon\mu)^+ = \nu_\epsilon$ and ν_ϵ is supported on X^+_ϵ. $\int_A \epsilon d\mu \le \lambda_2(A)$ for $A \subset X^+_\epsilon, A \in \Sigma$. This has to be trivial or $\mu(X^+_\epsilon) = 0$. But $\lambda_2 << \mu$. Therefore $\lambda_2(X^+_\epsilon) = 0$ and $\nu_\epsilon = 0$. This implies $\lambda_2(A) \le \epsilon\mu(A)$ for all A and $\epsilon > 0$. This means $\lambda_2 = 0$. The function f such that $\int_A f d\mu = \lambda(A)$ for all $A \in \Sigma$ is called the RN derivative and is denoted by $f = \frac{d\lambda}{d\mu}$.

Lemma. If $\lambda << \mu, f = \frac{d\lambda}{d\mu}$, g is integrable with respect to λ if and only if gf is integrable with respect to μ and

$$\int g d\lambda = \int f g d\mu$$

Proof. True if g is the indicator of a set. True if g is simple. True for bounded g. True for nonnegative g by taking sup over bounded nonnegative functions below it. Finally deal with g^+ and g^-.

Lemma. If $\lambda << \mu, \mu << \nu$ then $\lambda << \nu$ and $\frac{d\lambda}{d\nu} = \frac{d\lambda}{d\mu} \cdot \frac{d\mu}{d\nu}$

Proof. Let $\frac{d\lambda}{d\mu} = f$ and $\frac{d\mu}{d\nu} = g$. Then

$$\int h d\lambda = \int h f d\mu = \int h f g d\nu$$

This implies $\frac{d\lambda}{d\nu} = fg$.

Remark. The RN derivative is unique. $\int_A f d\mu = \int_A g d\mu$ for all $A \in \Sigma$ and f, g are Σ measurable then $f = g$ a.e. To see this note that $\int_A (f - g) d\mu = 0$ and we can take for A the set $\{x : f(x) - g(x) > 0\}$. Then $f - g \le 0$ a.e. A similar argument shows $f - g \ge 0$ a.e.. Implies $f = g$ a.e. If $\lambda << \mu$ on Σ and $\mathcal{S} \subset \Sigma$ is a sub $\sigma-$ field then $\lambda << \mu$ on $\mathcal{S}$ and the RN derivative will exist as an $\mathcal{S}$ measurable function that works only for $A \in \mathcal{S}$. Usually different from the RN derivative on Σ which is Σ measurable and works for $A \in \Sigma$.

We can consider measures that are infinite for the whole space. We will restrict our selves to σ-finite measures, which have the property that the whole space is a countable union of sets with finite measure. We consider simple functions $\sum_{j=1}^k c_j \chi_{A_J}(x)$ where each A_j is set of finite measure. The simple function is nonzero only on a set of finite measure. Any bounded measurable function that is nonzero only on a set of finite measure can be integrated as before. If all the functions are zero outside a fixed set of finite measure, bounded convergence theorem holds. For arbitrary nonnegative functions we define

$$\int f d\mu = \sup_{\substack{g:0\leq g\leq f \\ \mu[x:|g(x)|>0]<\infty}} \int g d\mu$$

It is easy to check that Fatou's lemma and the dominated convergence theorem hold good for $\sigma-$finite measures without any change.

A continuous function $F(x)$ on $(-\infty, \infty)$ is said to be of bounded variation if

$$C = \sup_{k, a_1 < a_2 < \cdots < a_k} \sum_{i=1}^{k-1} |F(a_j) - F(a_{j-1})| < \infty$$

Theorem. If F is of bounded variation then F can be written as $F(x) = c + F_1(x) - F_2(x)$ where F_1 and F_2 are bounded, nondecreasing, continuous functions with $F_1(-\infty) = F_2(-\infty) = 0$.

Proof. Given $\epsilon > 0$ there is some $k, a = a_1 < a_2 < \cdots a_k = b$ such that

$$\sum_{j=1}^{k-1} |F(a_{j+1}) - F(a_j)| \geq C - \epsilon$$

This forces $|F(x) - F(y)| < \epsilon$ for $x < y < a$ or $b < x < y$. In particular $\lim_{x\to-\infty} F(x) = F(-\infty)$ exists. We can take that as c. We can now assume that $F(-\infty) = 0$. Define

$$F_1(x) = \sup_{\substack{k, a_1 < a_2 < \cdots < a_k \\ \{a_j\} \leq x}} \sum_{j=1}^{k-1} (F(a_{j+1}) - F(a_j))^+$$

$$F_2(x) = \sup_{\substack{k, a_1 < a_2 < \cdots < a_k \\ \{a_j\} \leq x}} \sum_{j=1}^{k-1} (F(a_{j+1}) - F(a_j))^-$$

Since both x^+ and x^- are subadditive we can assume that the partitions are the same for both and $a_k = x$ and a_1 is close to $-\infty$ that $F_i(a_1) \leq \epsilon$. For $i = 1, 2$

$$F_i(x) - \sum_{j=1}^{k-1} (F(a_{j+1}) - F(a_j))^\pm \leq \epsilon$$

Taking the difference

$$|F(x) - (F_1(x) - F_2(x))| \leq F(a_1) + F_1(a_1) + F_2(a_1)$$

is as small as we want.

If f is integrable on $(-\infty,\infty)$ and $F(x) = \int_{-\infty}^{x} f(y)dy$ is $F'(x) = f(x)$? In Calculus it was proved for f continuous is it true in some sense for integrable f?

To prove results that are special about measures on R we need to understand the special relation between Borel sets and open sets.

Lemma. For any measure μ on a $\sigma-$field Σ generated by a field $\mathcal{F}$, given any $A \in \Sigma$ and any $\epsilon > 0$, there is a $B \in \mathcal{F}$ such that $\mu(A\Delta B) < \epsilon$. $(A\Delta B = (A \cap B^c) \cup (A^c \cup B))$.

Proof.. The class of A's for which this is true is a monotone class that contains the field.

Theorem. Given a set $A \in \mathcal{B}(R)$, a measure μ and $\epsilon > 0$, there is an open set $G \supset A$ and a closed set $C \subset A$ such that $\mu(G\backslash A) < \epsilon, \mu(A\backslash C) < \epsilon$.

Proof. Let us look at the collection of sets $\mathcal{A}$ for which we can do it. If A is closed $G_n = \{x : |x - y| < \frac{1}{n}\}$ decreases to A. n large will do it. The class of sets is closed under finite unions. Take the unions of G's and C's. The errors just add up. Complement is automatic because the definition is symmetric. Only need to show it is a monotone class. Given A_j pick G_j and C_j so that $\mu(G_j\backslash A_j) < \epsilon 2^-j$ and $\mu(A_j\backslash C_j) < \epsilon 2^{-(j+1)}$. We can take $G \supset \cup_j A_j \supset C$ with $G = \cup_j G_j$ and $C = \cup_j C_j$. But C is not closed. We can however replace C by $\cup_{j=1}^{N} C_j$ with N large enough that $\mu(C\backslash C_N) < \frac{\epsilon}{2}$. $G \supset \cup_j A_j \supset C_N$ works.

Theorem. Given a bounded measurable function f, there is a sequence $\{f_n\}$ of continuous functions such that $f_n \to f$ in measure.

Proof. Approximate it first by simple functions. It is enough to show that for any $A \in \Sigma$, $\chi_A(x)$ can be approximated by a continuous function such that $\mu[\{x : f(x) \neq \chi_A(x)\}] < \epsilon$ Given ϵ find an open set G and a closed set C such that $G \supset A \supset C$ and $\mu(G\backslash C) < \epsilon$. C and G^c are disjoint closed sets and we can construct a continuous function $g(x)$, $0 \leq g(x) \leq 1$, $g(x) = 1$ on C and 0 on G^c.

$$g(x) = \frac{d(x, G^c)}{d(x, G^c) + d(x, C)}$$

where $d(x, B) = \inf_{y \in B} |x - y|$

Theorem. Let $f \geq 0$ be integrable with respect to Lebesgue measure on R. Let

$$F(x) = \int_{-\infty}^{x} f(y)dy$$

Then F is nondecreasing, and with

$$f_h(x) = \frac{F(x + h) - F(x)}{h}$$

$f_h(x) \to f(x)$ as $h \to 0$ for almost all x, and $\lim_{h\to 0} \int_{-\infty}^{\infty} |f_h(x) - f(x)|dx = 0$.

Proof. Given $\epsilon > 0$, there is a function g, continuous and 0 outside a bounded interval $[a, b]$ such that $f(y) = g(y) + k(y)$ and

$$\int_{-\infty}^{\infty} |k(y)|dy < \epsilon$$

and

$$f_h(x) = g_h(x) + k_h(x)$$

It is clear that $g_h(x)$ converges uniformly to $g(x)$ and is 0 outside a fixed finite interval. We can estimate for $h > 0$,

$$\begin{aligned}
\int_{-\infty}^{\infty} |k_h(x)|dx &\leq \frac{1}{h}\int_{-\infty}^{\infty}\left[\int_{x}^{x+h}|k(y)|dy\right]dx \\
&= \frac{1}{h}\int_{-\infty}^{\infty}[\int_{y-h}^{y}dx]|k(y)|dy \\
&= \int_{-\infty}^{\infty}|k(y)|dy \\
&< \epsilon
\end{aligned}$$

Finally to prove convergence a.e. we need a lemma.

Vitali Covering Lemma. A collection of intervals $\mathcal{I}$ is said be a Vitali cover for a measurable set E, if given any $x \in E$, and $\epsilon > 0$, there is an $I \in \mathcal{I}$ such that $x \in I$ and $l(I) \leq \epsilon$

Lemma. Given a Vitali covering $\mathcal{I}$ of a set E of finite measure, and $\epsilon > 0$ there are disjoint intervals $I_1, \ldots, I_N \in \mathcal{I}$ such that $\mu[E \cap (\cup_{j=1}^{N} I_j)^c] < \epsilon$.

Proof. Take an open set G that contains E and has finite measure. We can assume that every $I \in \mathcal{I}$ is contained in G. We choose sequentially intervals $I_1, \ldots, I_n, \ldots$. Unless $E \subset \cup_{j=1}^{n} I_j$, after I_n, I_{n+1} is chosen so that its length is at least $\frac{k_n}{2}$ where k_n is the supremum of the lengths of all the intervals in $\mathcal{I}$ that do not meet $I_1, \ldots, I_n$. Since $\{I_j\}$ are disjoint and contained in G, $\sum_j l(I_j) < \infty$ and we can find N such that $\sum_{j=N+1}^{\infty} l(I_j) < \frac{\epsilon}{5}$. Let $R = E \cap (\cup_{j=1}^{N} I_j)^c$. Let $x \in R$. There is an interval I with $l(I)$ small enough containing x and disjoint from $\cup_{j=1}^{N} I_j$. If $I \cap I_n = \emptyset$, then $l(I) \leq k_n \leq 2l(I_{n+1})$. Since $l(I_{n+1}) \to 0$. I must meet one of the intervals $\{I_j\}$ say I_n. The distance from x to the mid point of I_n is $l(I) + \frac{1}{2}l(I_n) \leq \frac{5}{2}l(I_n)$. If we blow up I_n by a factor of 5 keeping the center and call the interval J_n, $R \subset \cup_{n=N+1}^{\infty} J_n$ and $\mu(R) \leq \epsilon$.

We define for any nondecreasing bounded function $f(x)$ on $[a, b]$ the derivatives

$$(D_+^+ f)(x) = \limsup_{\substack{h\to 0 \\ h>0}} \frac{f(x+h) - f(x)}{h}$$

$$(D_+^- f)(x) = \liminf_{\substack{h\to 0 \\ h>0}} \frac{f(x+h) - f(x)}{h}$$

$$(D_-^+ f)(x) = \limsup_{\substack{h\to 0 \\ h<0}} \frac{f(x+h) - f(x)}{h}$$

$$(D^-_- f)(x) = \liminf_{\substack{h\to 0\\ h<0}} \frac{f(x+h)-f(x)}{h}$$

are defined on (a,b).

Theorem. For almost all x the four derivatives are finite and equal. The derivative g is nonnegative, integrable and

$$f(b)-f(a) \geq \int_a^b g(x)dx$$

Proof. It is enough to show that for any two rationals $u < v$

$$\mu[x : (D^+_+ f)(x) < u < v < (D^+_- f)(x)] = 0$$

Let $E = E_{u,v}$ be the set in question. Assume $\mu(E) = s > 0$. Find an open set $G \supset E$ with $\mu[G \backslash E] < \epsilon$. For each $x \in E$ we can find h as small as we please such that the intervals are contained in the open set G and $f(x+h) - f(x) < uh$. By Vitali covering lemma we can find disjoint intervals $I_r = [x_r, x_r + h_r]$ such that their union A covers E to within ϵ. Each $x \in A$ is the end point of $[x-k, x]$ with $f(x) - f(x-k) > vk$ and contained in some I_r. We can find intervals$J_i = [y_i - k_i, y_i]$ with $f(y_i - k_i -) f(y_i) > vk$ their union covers A to within measure ϵ. Each J_i is contained in some I_r. Since f is increasing the sum over the J_i must be less than the sum over I_r.

$$(s-2\epsilon)v \leq (s+\epsilon)u$$

Contradicts $u < v$. Fatou's lemma shows

$$\begin{aligned}\int_a^b g(x)dx &\leq \liminf_{n\to\infty} n \int_a^b [f(x+\frac{1}{n}) - f(x)]dx \\ &= \liminf_{n\to\infty} \left[n\int_b^{b+\frac{1}{n}} f(x)dx - n\int_a^{a+\frac{1}{n}} f(x)dx \right] \\ &\leq f(b) - f(a)\end{aligned}$$

Given $F(x)$ that is continuous and nondecreasing on $[0,1]$ we know $F'(x) = f(x)$ exists a.e and is integrable with $\int_0^1 f(x)dx \leq F(1) - F(0)$. When does equality hold?

Equivalently when does the measure λ corresponding to F absolutely continuous with respect to Lebesgue measure.

A continuous and nondecreasing function $F(x)$ is absolutely continuous if for any $\epsilon > 0$ there is a $\delta > 0$ such that when ever $\sum_{i=1}^n (b_i - a_i) < \delta$, where $\{(a_i, b_i)\}$ are any finite collection of disjoint intervals in $[0,1]$ we have $\sum_{i=1}^n (F(b_i) - F(a_i)) < \epsilon$.

What we need to show if for any $\epsilon > 0$ there is a $\delta > 0$ such that for any set $A \in \mathcal{F}$ with $\mu(A) < \delta$ we have $\lambda(A) < \epsilon$, then $\lambda << \mu$ on Σ generated by $\mathcal{F}$. We can choose A_n from $\mathcal{F}$ approximating A under both λ and μ. Eventually $\mu(A)$ will get smaller than any δ forcing $\lambda(A)$ to be small and eventually 0.

Absolutely continuous functions are the indefinite integrals of Lebesgue integable functions.

Product Measures and Fubini's Theorem.

Let (X_i, Σ_i, μ_i) for $i = 1, 2$ be two measure spaces and we define (X, Σ, μ), the product of the two as follows.

$X = X_1 \times X_2$ is the cartesian product. Sets of the form $A_1 \times A_2 = \{(x_1, x_2) : x_1 \in A_1, x_2 \in A_2$ with $A_i \in \Sigma_i$ are called rectangles and they form a semiring. Finite disjoint unions form a field and the $\sigma-$field generated by it is $\Sigma = \Sigma_1 \times \Sigma_2$. We define the product measure by $\mu(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2)$ and show that it extends uniquely as a countably additive measure on $\Sigma_1 \times \Sigma_2$. We need to show that if R_j are disjoint rectangles $A^1_j \times A^2_j$ and their union is a rectangle $A^1 \times A^2$ then

$$\mu_1(A^1) \cdot \mu_2(A^2) = \sum_{j=1}^{\infty} \mu_1(A^1_j) \cdot \mu_2(A^2_j)$$

What we have is

$$\chi_{A^1}(x_1)\chi_{A^2}(x_2) = \sum_{j=1}^{\infty} \chi_{A^1_j}(x_1)\chi_{A^2_j}(x_2)$$

We can integrate x_2 term by term with respect to μ_2. Using the Bounded convergence theorem we have for each x_1

$$\chi_{A^1}(x_1)\mu_2(A^2) = \sum_{j=1}^{\infty} \chi_{A^1_j}(x_1)\mu_2(A^2_j)$$

Now integrate x_1 with respect to μ_1 and we have what we need.

We denote by $\mu = \mu_1 \times \mu_2$ the product measure. Fubini's Theorem asserts that if $f(x_1, x_2)$ is integrable with respect to μ then for almost all x_1 with respect to μ_1 it is integrable in x_2 with respect to μ_2 and the integral $g_1(x_1)$ is integrable with respect to μ_1. Moreover

$$\int_X f(x_1, x_2) d\mu = \int_{X_1} [\int_{X_2} f(x_1, x_2) d\mu_2] d\mu_1 = \int_{X_2} [\int_{X_1} f(x_1, x_2) d\mu_1] d\mu_2$$

True for indicator rectangles. True for indicators of sets in $\mathcal{F}$. True for indicators of sets in Σ. True for simple functions. True for bounded measurable functions. True for nonnegative functions and finally integrable functions.

Warning. Measurability in X is important. (joint measurability). There are crazy examples of sets in X such that for the indicator f

$$1 = \int_{X_1} [\int_{X_2} f(x_1, x_2) d\mu_2] d\mu_1 \neq \int_{X_2} [\int_{X_1} f(x_1, x_2) d\mu_1] d\mu_2 = 0$$

For nonnegative jointly measurable functions if any of the repeated integrals is finite then the double integral is finite as well.