Metric Spaces. (X, d) is a metric space if X is provided with a metric $d : X \times X \to R$ with the following properties.

1. $d(x, y) = d(y, x)$ for all $x, y \in X$

2. $d(x, y) \geq 0$ $d(x, y) = 0$ if and only if $x = y$.

3. For $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle Inequality)

Definition. $x_n \to x$ in X if $d(x_n, x) \to 0$. i.e given any $\epsilon > 0$ there is n_0 such that $d(x_n, x) < \epsilon$ for $n > n_0$.

Definition $\{x_n\}$ is a Cauchy sequence if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. i.e given ϵ there is n_0 such tat $d(x_n, x_m) \leq \epsilon$ for $n, m \geq n_0$.

A convergent sequence is Cauchy. $d(x_n, x_m) \leq d(x_n x) + d(x_m, x)$.

If a subsequence x_{n_j} of a Cauchy sequence of x_n converges to a limit x then the entire sequence converges to it.

$$d(x_n, x) \leq d(x_{n_j}, x) + d(x_{n_j} . x_n)$$

X is *complete* if every Cauchy sequence converges to a limit.

Theorem. If (X, d) is not complete, there is a complete space (Y, D) such that there is an embedding $y = Tx$ of X into Y such that $d(x_1, x_2) = D(Tx_1, Tx_2)$ and TX is dense in Y. Such a (Y, D) is unique up to isometry, i.e. if $(Y_1, D_1), (Y_2, D_2)$ are two choices then there is a one to one map U from Y_1 to Y_2 that is onto and $D_1(y_1, y_2) = D_2(Uy_1, Uy_2)$. (Y, D) is called the completion of (X, d).

Proof. Consider the space $\mathcal{Z}$ of all Cauchy sequences $\xi = \{x_n\}$ from (X, d). We define the distance
$$D(\xi, \eta) = \lim_{n\to\infty} d(x_n, y_n)$$
It is easy to check using triangle inequality that $|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n + y_m)$. Since R is complete the Cauchy sequence $d(x_n, y_n)$ has a limit. Note that this limit is also $\lim_{n,m\to\infty} d(x_n, y_m)$. It is possible that $D(\xi, \eta) = 0$. In that case we say they belong to the same equivalence class. The triangle inequality provides transitivity and we have symmetry. $\mathcal{Z}/\sim$ is taken as the completion of (Y, D) of (X, d). X is imbedded in Y by sending x to the equivalence class of all sequences that converge to x in particular the equivalence class containing $\xi_x = \{x, x, x, \ldots, x, \ldots\}$. X is imbedded densely because if $\{x_n\}$ is a Cauchy sequence, the equivalence class of Cauchy sequences ξ_n that that converge to x_n converges to the equivalence class ξ containing $\{x_n\}$.

$$\lim_{n\to\infty} D(\xi_{x_n}, \xi) = \lim_{n\to\infty} \lim_{m\to\infty} d(x_n, x_m) = 0$$

Finally we need to prove that (Y, D) is complete. Let $\{\xi_i\} = \{x_{i,n}\}$ be a Cauchy sequence of Cauchy sequences with $D(\xi_i, \xi_j) \to 0$ as $i, j \to \infty$.

$$\lim_{i,j\to\infty} \lim_{n\to\infty} d(x_{i,n}, x_{j,n}) = 0$$

For each i we can choose n_i such that $d(x_{i,n}, x_{i,m}) \le 2^{-i}$ for $n, m \ge n_i$. Consider the sequence $x_i = x_{i,n_i}$.

$$d(x_{i,n_i}, x_{j,n_j}) \le d(x_{i,n_i}, x_{i,m}) + d(x_{i,m}, x_{j,m}) + d(x_{j,m}, x_{j,n_j})$$

We can let $m \to \infty$.

$$d(x_{i,n_i}, x_{j,n_j}) \le d(x_{i,n_i}, x_{i,m}) + d(x_{i,m}, x_{j,m}) + d(x_{j,m}, x_{j,n_j})$$

$$d(x_{i,n_i}, x_{j,n_j}) \le 2^{-i} + D(\xi_i, \xi_j) + 2^{-j}$$

Makes $\eta = \{x_{i,n_i}\}$ a Cauchy sequence. We now show that $D(\xi_k, \eta) \to 0.$ as $k \to \infty$

$$D(\xi_i, \eta) = \lim_{k,\ell \to \infty} d(x_{i,\ell}, x_{k,n_k}) \le \lim_{k,\ell \to \infty} D(\xi_i, \xi_k) + 2^{-i} + 2^{-k}$$

and

$$\lim_{i \to \infty} \lim_{k,\ell \to \infty} d(x_{i,\ell}, x_{k,n_k}) = 0$$

Examples of Metric spaces.

1. $X = R$, $d(x, y) = |x - y|$

2. $X = R^n$. $d(x, y) = \sqrt{\sum (x_i - y_i)^2}$

3. $X = R^n$. $d(x, y) = \sum_i |x_i - y_i|$ or $d(x, y) = \sup_i |x_i - y_i|$

4. $X = L_1[0, 1]$. Lebesgue integrable functions $f(\cdot)$ on $[0, 1]$. $d(x, y) = \int_0^1 |f(x) - g(x)| dx$

5. $X = C[0, 1]$ Continuous functions $f(\cdot)$ on $[0, 1]$. $d(x, y) = \int_0^1 |f(x) - g(x)| dx$

6. $X = C[0, 1]$ Continuous functions $f(\cdot)$ on $[0, 1]$. $d(x, y) = \sup_x |f(x) - g(x)|$

7. Lebesgue mesurable functions $f(\cdot)$ on $[0, 1]$ such that $|f(x)|^p$ is integrable. $(1 \le p < \infty)$. $d(f, g) = \big[\int_0^1 |f(x) - g(x)|^p dx \big]^{\frac{1}{p}}$

Triangle inequality for L_p. Minkowski Inequality. Holder Inequaity. If $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \ge 1$

$$\int |f(x)g(x)| d\mu \le [\int |f(x)|^p d\mu]^{\frac{1}{p}} [\int |g(x)|^q d\mu]^{\frac{1}{q}}$$

$$(\int |f(x)|^p d\mu)^{\frac{1}{p}} = \sup_{g : \int |g(x)|^q d\mu \le 1} |\int f(x)g(x) d\mu|$$

$$\begin{aligned}
(\int |f_1(x) + f_2(x)|^p d\mu)^{\frac{1}{p}} &= \sup_{g : \int |g(x)|^q d\mu \le 1} [\int |(f_1(x) + f_2(x))g(x)| d\mu] \\
&\le \sup_{g : \int |g(x)|^q d\mu \le 1} \int |f_1(x)g(x)| d\mu + \sup_{g : \int |g(x)|^q d\mu \le 1} \int |f_2(x)g(x)| d\mu \\
&= (\int |f_1(x)|^p d\mu)^{\frac{1}{p}} + (\int |f_2(x)|^p d\mu)^{\frac{1}{p}}
\end{aligned}$$

Step 1. Let x, y be nonnegative. $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$xy \le \frac{x^p}{p} + \frac{y^q}{q}$$

Proof. Calculate $\sup_y [xy - \frac{y^q}{q}]$. Setting the derivative with respect to y as 0, $x = y^{q-1}$ or $y = x^{\frac{1}{q-1}} = x^{\frac{p}{q}}$

$\sup_y [xy - \frac{y^q}{q}] = x^{1+\frac{p}{q}} - \frac{x^p}{q} = x^p(1 - \frac{1}{q}) = \frac{x^p}{p}$. Proves the inequality.

Step 2.

$$\int |f(x)g(x)| d\mu \le [\int |f(x)|^p d\mu]^{\frac{1}{p}} [\int |g(x)|^q d\mu]^{\frac{1}{q}}$$

Proof. For any $\lambda > 0$

$$\begin{aligned} \int |f(x)g(x)| d\mu &= \int |(\lambda f(x))(\frac{g(x)}{\lambda})| d\mu \\ &\le \int [\frac{\lambda^p |f(x)|^p}{p} + \frac{|g(x)|^q}{q\lambda^q}] d\mu \end{aligned}$$

Minimize over $\lambda > 0$. $\lambda = (\int |g(x)|^q d\mu)^{\frac{1}{p+q}} (\int |f(x)|^p d\mu)^{-\frac{1}{p+q}}$

$$\lambda^p \int |f(x)|^p d\mu = \lambda^{-q} \int |g(x)|^q d\mu = (\int |f(x)|^p d\mu)^{\frac{1}{p}} (\int |g(x)|^q d\mu)^{\frac{1}{q}}$$

and

$$\frac{1}{p} + \frac{1}{q} = 1$$

Step 3. Assume that $\int |f(x)|^p d\mu < \infty$. Clearly

$$(\int |f(x)|^p d\mu)^{\frac{1}{p}} \ge \sup_{g: \int |g(x)|^q d\mu \le 1} |\int f(x)g(x) d\mu|$$

Take $g(x) = c(sign(f(x))|f(x)|^{p-1}$ where

$$\int |g(x)|^q d\mu = c^q \int |f(x)|^p d\mu = 1$$

Then

$$c \int f(x)g(x) d\mu = c \int |f(x)|^p d\mu$$

and

$$c = (\int |f(x)|^p d\mu)^{-\frac{1}{q}}$$

$$c\int |f(x)|^p d\mu = (\int |f(x)|^p d\mu)^{1-\frac{1}{q}} = (\int |f(x)|^p d\mu)^{\frac{1}{p}}$$

It then follows that

$$(\int |f(x)+g(x)|^p)^{\frac{1}{p}} \leq (\int |f(x)|^p)^{\frac{1}{p}} + (\int |g(x)|^p)^{\frac{1}{p}}$$

Step 4. Actually if for a measurable function f

$$\sup_{\substack{g:\int |g|^q d\mu=1 \\ g\in\mathcal{G}}} \int f(x)g(x)d\mu < \infty$$

where $\mathcal{G}$ consists of functions g that are bounded and supported on some set E of finite measure on which f is bounded as well, then $\int |f|^p d\mu < \infty$ and the Step 3 is applicable. Replace f by $f\chi_E$ and and get a bound for $\int |f_E(x)|^p d\mu$ that is uniform in E.