Closed and Open Sets. Aset C in a metric space is closed if whenever $x_n \in C$ and $d(x_n, x) \to 0$ the limit $x \in C$.

Finite union of closed sets is closed. Arbitrary intersection of closed sets is closed. (an infinite union of closed sets need no be closed.)

From $x_n \in (C_1 \cup C_2 \cdots \cup C_k)$ for $n \geq 1$ it follows that an infinite number of the sequence must belong to some C_i. Since the subsequence converges to the sam limit x, $x \in C_i$ and therefore $x \in (C_1 \cup C_2 \cdots \cup C_k)$. For arbitrary intersections $C = \cap_\alpha C_\alpha$, since $x_n \in C_\alpha$ for each α the limit $x \in C_\alpha$ and thus $x \in C = \cap_\alpha C_\alpha$. Finally a single point in R is a closed set. But the rationals, a countable union of single points is not closed.

Let (X, d) be a complete metric space. The metric d restricted to $A \subset X$ is complete if and only if A is closed. Cauchy sequences in (A, d) are Cauchy sequences in (X, d). They converge in X because X is compete.Te question is do they converge in (A, d). Only if $x \in A$. This happens for all sequences that converge in (X, d) if and only if A is closed.

An open set is the complement of a closed set. An intersection of a finite number of open sets is open. Arbitrary union of open sets is open. These two facts are proved by complementation.

$D(x, \epsilon) = \{y : d(x, y) < \epsilon\}$ are open. If $y_n \in D(x, \epsilon)^c$ and $y_n \to y$, then $d(x, y_n) \geq \epsilon$ and $d(x, y) \leq d(x, y_n) + d(y, y_n)$. Letting $n \to \infty$ we get $d(x, y) \geq \epsilon$. A set G is open if and only if around given any $x \in G$ there in an $\epsilon = \epsilon_x$ such that $D(x, \epsilon) \subset G$. If so G is a union of such discs and is a union of open sets. So it is open. If not there is an x such that for any $\epsilon > 0$, $D(x, \epsilon) \cap G^c \neq \emptyset$. This will provide a sequence from G^c that will converge to $x \in G$, making G^c not closed. So G is not open.

A collection of open sets is a basis if any open set can be expressed as the union of a subset of opens sets from the collection. Equivalently a collection $\{U_\alpha\}$of open sets is basis if given any open set G and an point $x \in G$, there some U_α such that $x \in U_\alpha \subset G$. The discs $\{d(x, \epsilon)\}$ as x and ϵ vary is a basis. The radii ϵ can be limited to a sequence $\epsilon_k \to 0$. the centers $\{x\}$ can be limited to a countable dense set. Given $x \in G$ a point in an open set there is an ϵ such that $d(x, \epsilon) \subset G$. If $\{x_k\}$ is a dense set then there is a k such that $d(x, x_k) < \frac{\epsilon}{2}$. Then $x \in D(x_k, \frac{\epsilon}{2}) \subset D(x, \epsilon) \subset G$.

A metric space (X, d) is separable if there is a countable dense subset. This is equivalent to having a countable basis of open sets. We saw that if $\{x_i\}$ is dense then $D(x_i, \frac{1}{k})$ is a basis. Conversely if $\{U_i\}$ is any countable basis then if we choose any $x_i \in U_i$, $\{x_i\}$ is dense in X. Consider $D(x, \epsilon)$ for any $x \in G$ and arbitrary small ϵ. Being open this is union of some sub-collection of $\{U_i\}$. In particular it must contain some U_i. The point x_i chosen from that U_i is with in a distance ϵ from x.

If X is a separable metric space and A is any subset, any covering of A by open sets has a countable sub-cover. Let $\cup_\alpha G_\alpha$ be a cover and $\{U_i\}$, $i \in I$ a countable basis. Each G_α is a union of some U_i. Let J_α be the indices involved.Then A is covered by $J = \cup_\alpha J_\alpha \subset I$. Since I is countable we can pick α_i for each $i \in J$ and it will give us a countable sub-cover.

If $A \subset X$ is an subset then (A, d) is a metric space and its closed (or open) subsets are precisely the closed (or open) subsets of (X, d) intersected with A. If C is closed in X, then $C \cap A$ is closed in A. If $x_n \in C \cap A$ and $d(x_n, x) \to 0$, and $x \in A$, since $x_n \in C$ and $x_n \to x$ and C is closed $x \in C$. Therefore $x \in C \cap A$, making it closed. Conversely if C is closed in A then $\overline{C}$ the closure of C in X is closed in X and $\overline{C} \cap A = C$.

Compactness. A set $A \subset X$ is compact if every sequence $\{x_n\}$ from A has a subsequence that converges to a limit $x \in A$. A compact set is necessarily closed. A set is compact if and only if every open covering has a finite subcover.

A compact metric space is separable. To see this for each k consider a maximal collection F_k of points in X such that the discs $\{D(x, \frac{1}{k})\}$ centered at $x \in F$ of radius $\frac{1}{k}$ are disjoint. It has to be finite for otherwise the sequence of their centers will not have a convergent subsequence. The union $\cup_k F_k$ is a countable set. We claim that $\cup_{x \in F_k} D(x, \frac{2}{k})$ is X. Any missed point y will have the property that $D(y, \frac{1}{k}) \cap D(x, \frac{1}{k}) = \emptyset$ implying that F_k is not maximal. Clearly $\cup_k F_k$ is dense.

Since every open covering has a countable sub cover it is enough to show that any countable cover has a finite subcover, Let $\cup_i G_i$ be a cover but for any n, $\cup_{i=1}^n G_i$ is not. Let us pick x_n from $[\cup_{i=1}^n G_i]^c = C_n$. $C_1 \supset C_2 \cdots C_n \cdots$. For $j \geq n$ $x_j \in C_n$ and if a subsequene of x_n has a limi x then $x \in C_n$ for every n contradicts $X = \cup_{i=1}^{\infty} G_i$ being a cover.

A limit point of a sequence is a point x such that every neighborhood $D(x, \epsilon)$ contains an infinite number of members of the sequence. Then a subsequence will converge to x. Conversely if a subsequence converges to x then every neighborhood $D(x, \epsilon)$ contains an infinite number of members of the sequence. If X has a sequence with no limit points, every point $x \in X$ will have a disc $D(x, \epsilon_x)$ that contains only a finite number of members from the sequence, As x varies over X is a cover. Take a finite subcover. It contains only a finite number of members from the sequence. Contradiction.

Finite intersection property. In a compact space if a family of closed sets have the property that every finite intersection is nonempty then the whole intersection is nonempty. Just by taking complements seen as the same as the open covering property.

$A \subset X$ is compact if the metric restricted to A makes A compact. This means every sequence from A must have a subsequence converging to a point in A. In particular A has to be closed. Any closed subset of a compact metric space is compact. A compact metric space is complete.

Baire Category Theorem. If X is a complete metric space and it is a countable union of closed sets $\{C_i\}$ at least one of them must have an interior, i.e. an open set $U \subset C_i$. [Interior is the largest open set contained in a set. i.e. complement of the closure of the complement.] Equivalently the intersection $A = \cap G_i$ of a countable set of dense open sets is dense.

Proof. Suppose it is not. Then there is a disc $D(x, \epsilon)$ that does not intersect A. Since G_1 is dense there is an x_1 from G_1 inside $D(x, \epsilon)$. Since G_1 is open there is $\epsilon_1 > 0$ such that $D(x_1, \epsilon_1) \subset \overline{D(x_1, \epsilon_1)} \subset G_1$. Proceeding inductively we have a sequence of discs $D(x_n, \epsilon_n)$ such that $\epsilon_n \to 0$, $D(x_n, \epsilon_n) \subset G_n$ and

$$D(x_n, \epsilon_n) \subset \overline{D(x_n, \epsilon_n)} \subset D(x_{n-1}, \epsilon_{n-1})$$

It is easy to see that the sequence $\{x_n\}$ is a Cauchy sequence. If $n > m$, $d(x_n, x_m) < \epsilon_m$. It has a limit. $x_n \in \overline{D(x_j, \epsilon_j)}$ for $n \geq j$ and so must the limit x of x_n. But

$$\overline{D(x_{j+1}, \epsilon_{j+1})} \subset D(x_j, \epsilon_j) \subset G_j$$

$x \in G_i$ for every i, $x \in A$. There is no open set disjoint from A. Makes A dense.

Continuous Functions.

(X, d) and (Y, D) are two metric spaces. A map or a function $y = F(x)$ is continuous at x if the following equivalent things are true.

1. If $d(x_n, x) \to 0$, then $D(F(x_n), F(x)) \to 0$.

2. Given any $\epsilon > 0$ there is a $\delta > 0$ such that $D(F(y), F(x)) < \epsilon$ if $d(x, y) < \delta$

3. If U is any open set containing $F(x)$, then $F^{-1}(U) = \{x : F(x) \in U\}$ contains an open set that contains x.

A continuous function is one continuous at every point. In that case (**3.**) can be replaced by

3a The inverse image $F^{-1}(U)$ of every open set U in Y is open in X

3a The inverse image $F^{-1}(U)$ of every open set in Y is open in X **3b** The inverse image $F^{-1}(C)$ of every closed set C in Y is closed in X

Proof. Let (**1**) hold. If (**2**) does not hold, there is a positive ϵ_0 for which there is no δ. For any $\delta > 0$ there is a y such that $d(x, y) < \delta$ but $D(F(y), F(x)) \geq \epsilon_0$. This produces a sequence with $d(x_n, x) \to 0$ but $D(F(y), F(x)) \geq \epsilon_0$ contradicting (**1**).

(**2**) can be restated as $F^{-1}(D(F(x), \epsilon) \supset D(x, \delta)$ for some $\delta > 0$. U will contain some disc $D(F(x), \epsilon)$ and $F^{-1}(U) \supset D(x, \delta)$ for some $\delta > 0$. This is (**3**).

If (**3**) holds, for any $\epsilon > 0$, $F^{-1}D(x, \epsilon)$ contains $D(x, \delta)$ for some $\delta > 0$. If $d(x_n, x) \to 0$, then $x_n \in D(x, \delta)$ for large n, and this implies $x_n \in F^{-1}D(F(x), \epsilon)$ or $D(F(x_n), F(x)) \to 0$.

(**3a**) is just (**3**) at every x and (3b.) is just the same (**3a**) expressed for complements.

Continuous function of a continuous function is continuous.

A continuous image of a compact set is compact. $A \subset X$ is compact. $F : X \to Y$ is a continuous map. $B = F(A)$ is a compact subset of Y. If $y_n \in B$ then $y_n = F(x_n)$ for some $x_n \in A$. Since A is compact there is a subsequence $\{x_{n_j}\}$ that converges to a limit x in A. $F(x_{n_j})$ will converge to $F(x) \in B$.

In particular a continuous real valued function on a compact space is bounded.

Product space. $X = X_1 \times X_2$.

$$d_X((x_1, x_2), (y_1, y_2)) = d_{X_1}(x_1, y_1) + d_{X_2}(x_2, y_2)$$

defines a metric. If X_1, X_2 are complete so is X. If X_1, X_2 are compact so is X.

Convergence does not need a metric. Needs only the collection $\mathcal{T}$ of open sets, closed under arbitrary unions and finite intersections, that contains X and $\emptyset$.

A neighborhood of x is an open set U containing x. $x_n \to x$ if given any such U, $x_n \in U$ for n large enough. U replaces the disc. We are interested in determining if there is a metric on X under which $\mathcal{T}$ are precisely the open sets.