If $G \in X$ is an open subset of a complete metric space X, (G, d) is not complete unless G is also closed. But we can change he metic so that (G, d_1) and (G, d) have the same open sets, i.e $d(x_n, x) \to 0$ if and only if $d_1(x_n, x) \to 0$ provided $x_n, x \in G$. But (G, d_1) is complete.

Define

$$d_1(x, y) = d(x, y) + \left|\frac{1}{d(x, G^c)} - \frac{1}{d(y, G^c)}\right|$$

If $x_n, x \in G$ and $x_n \to x$ $d_(x_n, G^c) \to d(x, G^c)$ and $d(x, G^c) > 0$ for $x \in G$. Therefore $d_1(x_n, x) \to 0$. $d_1 \geq d$. If we have a Cauchy sequence $\{x_n\}$, in d_1, $\frac{1}{d(x_n, G^c)}$ has a limit and therefore bounded keeping $d(x_n, G^c)$ away from 0, forcing the limit x under d to be in G. So every Cauchy sequence under d_1 converges in d_1 to a limit in G.

Metrization.

Let a space X and a collection $\mathcal{T}$ of subsets of X satisfying the properties below be given.

1. X and $\emptyset$ are in $\mathcal{T}$

2. $\mathcal{T}$ is closed under arbitrary union and finite intersection.

We are looking for a metric d on X such that (X, d) is a separable metric space and $\mathcal{T}$ is the collection of open sets in this metric. We denote by $\mathcal{C}$ the collection of sets that are complements of sets in $\mathcal{T}$ and these will be the collection of closed sets. We make the following assumptions on $(X, \mathcal{T})$ which are clearly necessay.

1. The set consisting of the single point x is closed for every $x \in X$.

2. $\mathcal{T}$ has a countable basis $\{U_j\}$ such that every open set i.e. set in $\mathcal{T}$ is the union of a sub collection from $\{U_j\}$.

3. Given two closed sets $C_1, C_2 \in \mathcal{C}$ with $C_1 \cap C_2 = \emptyset$ there are sets $G_1, G_2 \in \mathcal{T}$ with $C_1 \subset G_1$, $C_2 \subset G_2$ and $G_1 \cap G_2 = \emptyset$.

To see that **3** is valid in any metric space define for $A \subset X$

$$d(x, A) = \inf_{y \in A} d(x, y)$$

which is a continuous function of x for every A. If A is closed then $d(x, A) = 0 \Leftrightarrow x \in A$.

$$G_1 = \{x : d(x, C_1) < d(x, C_2)\}, \quad G_2 = \{x : d(x, C_2) < d(x, C_1)\}$$

works.

Spaces with properties **1,** and **3** are called Normal.

Lemma. Let $(X, \mathcal{T})$ be a Normal space. Let $C_0 \subset G_1$ with C_0 closed and G_1 open. Let Q be the set of diadics $t = \frac{i}{2^n}$, $0 < t < 1$. Then for $t \in Q$, there are open sets G_t such that if $s, t \in Q$, $0 < s < t < 1$,

$$C_0 \subset G_s \subset \overline{G_s} \subset G_t \subset G_1$$

Proof. First we will show that given a closed set C_0 and an open set $G_1 \supset C_0$ there is an open set $G_{\frac{1}{2}}$ such that

$$C_0 \subset G_{\frac{1}{2}} \subset \overline{G}_{\frac{1}{2}} \subset G_1$$

Because the space is normal, and C_0 and G_1^c, are disjoint closed sets there are disjoint open sets $G_{\frac{1}{2}}$ and U with $C_0 \subset G_{\frac{1}{2}}$, $G_1^c \subset U$, and $G_{\frac{1}{2}} \subset U^c \subset G_1$. Since U^c is closed, we have

$$C_0 \subset G_{\frac{1}{2}} \subset \overline{G}_{\frac{1}{2}} \subset G_1$$

We can now repeat the process and obtain

$$C_0 \subset G_s \subset \overline{G_s} \subset G_t \subset G_1$$

for all diadics.

Lemma. The function

$$f(x) = \{\inf s : x \in G_s\}$$

if $x \in G_1$ and $f(x) = 1$ otherwise is continuous $f(x) = 0$ 0n C_0 and $f(x) = 1$ on G_1^c.
Proof.

$$\{x : f(x) < a\} = \cup_{t<a} G_t$$

are open and

$$\{x : f(x) \leq a\} = \cap_{t>a} G_t = \cap_{s>a} \overline{G}_s$$

are closed. $f^{-1}(a,b) = \{x : f(x) < b\} \cap \{x : f(x) > a\}$ are open. Makes f continuous.

In a normal space given two disjoint closed sets C_1, C_2 there is a continuous function $f(x)$, $0 \leq f(x) \leq 1$ with $f(x) = 0$ on C_1 and 1 on C_2.

Urysohn Metrization Theorem. Let $(X, \mathcal{T})$ be Normal, with single points being closed sets, and having a countable basis for $\mathcal{T}$. Then there is a metric $d(x,y)$ such that $\mathcal{T}$ are precisely the open sets.

Proof. Let $\{G_i\}$ be a basis. A pair G_i, G_j is admissible if $\overline{G}_i \subset G_j$. For each such pair $\overline{G}_i$ and G_j^c are disjoint closed sets and there is a continuous function f that satisfies $0 \leq f(x) \leq 1$ and equals 0 and 1 on the two closed sets. We enumerate this countable collection into a single sequence $\{f_k\}$. Define

$$d(x,y) = \sum_{i=1}^{\infty} \frac{|f_k(x) - f_k(y)|}{2^k}$$

Clearly d is a distance. Since f_k are continuous and the series converges uniformly $d(x,y)$ is continuous and $d(x_n, x) \to 0$ if $x_n \to x$. We need to prove the converse. If $f_k(x_n) \to f_k(x)$ for some x, then $x_n \to x$.Given a neighborhood (open set) N we need to show that $x_n \in N$ for $n \geq n_0$. We can find a G_i from the basis that contains x and is contained in N. The point x is a closed set. There is an open set U such that

$$x \in U \subset \overline{U} \subset G_i$$

We can replace U by some G_j from the basis so that

$$x \in G_j \subset \overline{J}_j \subset G_i \subset N$$

We have a continuous function f_k which is 1 on N^c and 0 at x. if $f_k(x_n) \to 0$ then x_n must leave N^c for $n \geq n_0$.