Riesz Representation Theorem. Let $\Lambda(f)$ be a bounded linear functional on $C(X)$ the space of continuous functions on a compact metric space X.Then there is a signed measure μ on the Borel σ-field $\mathcal{B}$ of X, such that

$$\Lambda(f) = \int_X f(x) d\mu$$

This is done in several steps. Λ is non-negative if for every $f \geq 0$, $\Lambda(f) \geq 0$.

First we need a result called partition of unity. We will deal only with functions that satisfy $0 \leq f \leq 1$. We always assume it is so.

Lemma. Let X be compact metric space. Let $\{G_i\}$ be a finite collection open sets with $\cup_{i=1}^n G_i \supset C$ where C is a closed set. Then there are nonnegative continuous functions h_i with its support contained in G_i such that $\sum_{i=1}^n h_i = 1$ on C.

Proof. For each $x \in C$ there is some open set G_i that contains x, and therefore a ball $B(x, \delta(x))$ around x of radius $\delta(x)$ whose closure $\overline{B(x, \delta(x))}$ is contained in G_i. Such balls provide a covering of C and we extract a finite sub cover. Each ball is contained in some G_i and we divide them in to n groups depending on which G_i it is contained in. If there are several possibilities choose any one.. Let their unions be W_i with closures $\overline{W}_i \subset G_i$. There are functions g_i that are 1 on W_i with support contained in G_i. We define

$$h_1 = g_1, h_2 = g_2(1 - g_1), \cdots, h_n = g_n(1 - g_1) \cdots (1 - g_{n-1})$$

Then

$$h_1 + h_2 + \cdots + h_n = 1 - (1 - g_1) \cdots (1 - g_n)$$

Since some $g_i = 1$ at every point of C we are done.

1. Any bounded Λ can be written as $\Lambda^+ - \Lambda^-$ where $\lambda^\pm$ are both non-negative.

Proof. For $f \geq 0$, define

$$\Lambda^+(f) = \sup_{0 \leq g \leq f} \Lambda(g)$$

$$\Lambda^+(f_1 + f_2) = \Lambda^+(f_1) + \Lambda^+(f_2)$$

For $c > 0$

$$\Lambda^+(cf) = c\Lambda^+(f)$$

For arbitrary f we write $f = (f + C) - C$ and $\Lambda^+(f) = \Lambda^+(f + C) - \Lambda^+(C)$. It is well defined does not depend on C.

One defines $\Lambda^-(f) = \Lambda^+(f) - \Lambda(f)$ so that for $f \in C(X)$, $\Lambda(f) = \Lambda^+(f) - \Lambda^-(f)$. It is easy to verify that for $f \geq 0$, $\Lambda^-(f) \geq 0$ because $\Lambda^+(f) \geq \Lambda(f)$.

$$
\begin{aligned}
\|\Lambda^+\| + \|\Lambda^-\| &= \Lambda^+(1) + \Lambda^-(1) \\
&= \Lambda^+(1) + \Lambda^+(1) - \Lambda(1) \\
&= \sup_{0\le f\le 1} \Lambda(f) + \sup_{0\le g\le 1} \Lambda(g) - \Lambda(1) \\
&= \sup_{0\le f\le 1} \Lambda(f) + \sup_{0\le g\le 1} \Lambda(g-1) \\
&= \sup_{0\le f\le 1} \Lambda(f) + \sup_{0\le g\le 1} |\Lambda(-g)| \\
&= \sup_{\substack{0\le f\le 1\\ 0\le g\le 1}} \Lambda(f-g) = \sup_{0\le |f|\le 1} \Lambda(f) = \|\Lambda\|
\end{aligned}
$$

The problem now is reduced to proving that a non-negative linear functional which is bounded by $\Lambda(1)$ has the representation in terms of a non-negative measure μ.

$$\Lambda(f) = \int_X f(x)\, d\mu$$

2. For any open set G we define

$$\mu(G) = \sup_{\substack{0\le f\le 1\\ \text{support } f\subset G}} \Lambda(f)$$

Support f is $\overline{\{x : f(x) \neq 0\}}$.

Remark. We could take the sup over the larger class of f with $f = 0$ on G^c. Then $\{x : f(x) \le \epsilon\}$ will be a closed set in G. And $g = (f-\epsilon)^+$ with $\Lambda(g) \ge \Lambda(f) - \epsilon$ can replace f. The supremum will be the same.

3. For any set E we define

$$\mu(E) = \inf_{\substack{G\supset E\\ G \text{ open}}} \mu(G)$$

4 We say $E \in \Sigma$ if

$$\mu(E) = \sup_{\substack{C\subset E\\ C \text{ closed}}} \mu(C)$$

5. If $\{E_i\}$ is any countable collection of subsets of X

$$\mu(\cup_{i=1}^\infty E_i) \le \sum_{i=1}^\infty \mu(E_i)$$

Proof. Let us first show that if G_1, G_2 are open

$$\mu(G_1 \cup G_2) \le \mu(G_1) + \mu(G_2)$$

Given $\epsilon > 0$, there is a function $g_\epsilon(x)$, $0 \le g_\epsilon \le 1$, with support C_ϵ contained in $G_1 \cup G_2$ with $\Lambda(g_\epsilon) \ge \mu(G) - \epsilon$. There are two non negative functions h_1, h_2 with their supports contained in G_1 and G_2 with $h_1 + h_2 = 1$ on C_ϵ. $g_\epsilon = g_\epsilon h_1 + g_\epsilon h_2$.

$$\mu(G_1) + \mu(G_2) \ge \Lambda(g_\epsilon h_1) + \Lambda(g_\epsilon h_2) = \Lambda(g_\epsilon) \ge \mu(G_1 \cup G_2) - \epsilon$$

We can assume that $\sum_i \mu(E_i) < \infty$. Pick open sets $V_i \supset E_i$ such that $\mu(V_i) \le \mu(E_i) + \epsilon 2^{-i}$. Let $V = \cup_i V_i$. Let f be such that the support D of f is contained in V and $\Lambda(f) \ge \mu(V) - \epsilon$. Since D is compact and contained in V it is contained in $\cup_{i=1}^n V_i$ for some finite n.

$$\Lambda(f) \le \mu(\cup_{i=1}^n V_i) \le \sum_{i=1}^n \mu(E_i) + \epsilon \le \sum_{i=1}^\infty \mu(E_i) + \epsilon$$

Since this is true for every f with support contained in V

$$\Lambda(E) \le \Lambda(V) \le \sum_{i=1}^\infty \mu(E_i) + \epsilon$$

ϵ is arbitrary.

6. If C is a closed set then

$$\mu(C) = \inf\{\Lambda(f) : 0 \le f \le 1, f = 1 \text{ on } C\}$$

Proof. Since

$$\mu(C) = \inf\{\mu(V) : V \text{ open }; V \supset C\}$$

We need to show two things.

Given any f, such that $f = 1$ on C, for any $\epsilon > 0$, $\{x : f(x) > 1 - \epsilon\}$ is an open set $V_\epsilon \supset C$. If g is any function supported in V_ϵ, $(1-\epsilon)g \le f$ or $\Lambda(g) \le (1-\epsilon)^{-1}\Lambda(f)$. Since $\mu(V_\epsilon) = \{\sup \Lambda(g) : \text{support } g \subset V_\epsilon\}$ it follows that $\mu(V_\epsilon) \le (1-\epsilon)^{-1}\Lambda(f)$.

In the reverse direction given $V \supset C$ by Urysohn's lemma there is an f that is 1 on C with support inside V. Then $\Lambda(f) \le \mu(V)$.

7. If G is open

$$\mu(G) = \sup_{\substack{C \subset G \\ C \text{closed}}} \mu(C)$$

Proof. Since

$$\mu(G) = \sup\{\Lambda(g) : 0 \le g \le 1; \text{ support } g \subset G\}$$

We need to show two things.

G is an open set and $0 \le g \le 1$ is a function supported on a closed subset C of G. If $f = 1$ on C, then $f \ge g$ and $\Lambda(f) \ge \Lambda(g)$. If $W \supset C$ is any open set there is an f that is 1 on C and supported in W. Makes $\mu(W) \ge \Lambda(g)$. True for every $W \supset C$. Follows that $\mu(C) \ge \Lambda(g)$.

Conversely if $C \subset G$ is any closed subset of G, there is a function g, $0 \le g \le 1$ with support contained in G and

$$\Lambda(g) \ge \mu(G) - \epsilon \ge \mu(C) - \epsilon$$

8. If $\{E_i\}$ are in Σ and pairwise disjoint $E = \cup_{i=1}^{\infty} E_i \in \Sigma$ and

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$$

Proof. Let C_1 and C_2 be closed sets that are disjoint. There is a function f, $0 \le f \le 1$, $f = 1$ on C_1 and 0 on C_2. Let g equal 1 on $C_1 \cup C_2$ with $\Lambda(g) \le \mu(C_1 \cup C_2) + \epsilon$. $\Lambda(gf) \ge \mu(C_1)$ and $\Lambda(g(1-f)) \ge \mu(C_2)$. Adding $\mu(C_1 \cup C_2) + \epsilon \ge \Lambda(g) \ge \mu(C_1) + \mu(C_2)$ Letting $\epsilon \to 0$, $\mu(C_1 \cup C_2) \ge \Lambda(g) \ge \mu(C_1) + \mu(C_2)$. We already have the other half.

Since $E_i \in \Sigma$ there are closed sets $D_i \subset E_i$ with $\mu(D_i) \ge \mu(E_i) - \epsilon 2^{-i}$. $\{D_i\}$ are pairwise disjoint as well.

$$\mu(E) \ge \mu(\cup_{i=1}^{n} E_i) \ge \mu(\cup_{i=1}^{n} D_i) = \sum_{i=1}^{n} \mu(D_i) \ge \sum_{i=1}^{n} [\mu(E_i) - \epsilon \sum 2^{-i}]$$

with $n \to \infty$ and $\epsilon \to 0$

$$\mu(E) \ge \sum_{i=1}^{\infty} \mu(E_i)$$

We have the other half. Easy to check that $E \in \mathcal{M}$. $G_i \supset E_i \supset D_i$, $\mu(G_i) - \mu(C_i) \le \epsilon 2^{-i}$.Then $\cup_{i=1}^{\infty} G_i \supset \cup_{i=1}^{\infty} E_i \supset \cup_{i=1}^{n} D_i$.

9. For any $E \in \Sigma$ and any $\epsilon > 0$ there is an open set G and a closed set C such that $C \subset E \subset G$ and $\mu(G - C) \le \epsilon$.

Proof. From our definitions we can find C and G such that $C \subset E$ and $E \subset G$ and

$$\mu(C) \ge \mu(E) - \frac{\epsilon}{2}; \quad \mu(G) - \mu(E) \le -\frac{\epsilon}{2}$$

$G = C \cup (G - C)$ is a disjoint union and both are in Σ. $\mu(G) = \mu(C) + \mu(G - C)$. Therefore $\mu(G - C) \le \epsilon$.

10. Σ is a Field.

Proof. If $E_1, E_2 \in \Sigma$, for any $\epsilon > 0$ can find C_1, C_2, G_1, G_2 such that $C_i \subset E_i \subset G_i$ and $\mu(G_i - C_i) < \frac{\epsilon}{2}$. $((G_1 \cup G_2) - (C_1 \cup C_2)) \subset ((G_1 - C_1) \cup (G_2 - C_2))$. $\mu((G_1 - C_1) \cup (G_2 - C_2)) \le \epsilon$. $(E_1 \cup E_2) \in \Sigma$. Similarly intersection and complementation.

11. Σ is sigma field and μ is a measure on Σ.

Proof. Done.

12. $\int f d\mu = \Lambda(f)$

Proof. It is enough to prove $\Lambda(f) \leq \int f d\mu$. We can add constants to both sides $\Lambda(1) = \mu(X)$. Can assume $f \geq 0$. Divide by a constant $0 \leq f \leq 1$.

Let $\epsilon > 0$ be given. Let $\{0 = y_0 < y_1 < \cdots < y_n = 1\}$ be the interval $[0,1]$ divided into n equal parts such that $\frac{1}{n} < \epsilon$. Let $E_i = \{x : y_{i-1} < f(x) \leq y_i\}$. We can include $f^{-1}(0)$ in E_1. E_i are disjoint sets, $X = \cup_i E_i$. There are open sets $G_i \supset E_i$ with $\mu(G_i) < \mu(E_i) + \frac{\epsilon}{n}$ and $f(x) \leq y_i + \epsilon$. Since $\{G_i\}$ is a covering of X, there is a partition of unity $\{h_i\}$ with $\sum_i h_i = 1$, and h_i supported inside G_i. We have $f = \sum_i h_i f$. Note that $\Lambda(h_i) \leq \mu(G_i) \leq \mu(E_i) + \frac{\epsilon}{n}$.

$$\Lambda(f) = \sum_{i=1}^{n} \Lambda(h_i f) \leq \sum_{i=1}^{n} (y_i + \epsilon)\Lambda(h_i) \leq \sum_{i=1}^{n} (y_i - \epsilon + 2\epsilon)[\mu(E_i) + \frac{\epsilon}{n}] + 2\epsilon$$
$$\leq \sum_{i=1}^{n} (y_i - \epsilon)\mu(E_i) + 2\epsilon + \epsilon(1+\epsilon) \leq \int f d\mu + 3\epsilon + \epsilon^2$$

Dual of L_p spaces. Let (Ω, Σ, μ) be a measure space where μ is a finite measure on the σ-field Σ of subsets of Ω. $\mathcal{X}$ be the Banach space $L_p(\Omega, \Sigma, \mu)$ of Σ measurable functions that satisfy $\int_\Omega |f(\omega)|^p d\mu < \infty$ with the norm

$$\|f\|_p = \left[\int_\Omega |f(\omega)|^p d\mu\right]^{\frac{1}{p}}$$

for $1 \leq p < \infty$. Let $\Lambda(f)$ be a bounded linear functional on $L_p(\Omega, \Sigma, \mu)$. If $1 < p < \infty$

$$\Lambda(f) = \int f g d\mu$$

for some $g \in L_q(\Omega, \Sigma, \mu)$ where $\frac{1}{p} + \frac{1}{q} = 1$. $\|\Lambda\| = \|g\|_q$. If $p = 1$, $q = \infty$ and it is still true that

$$\Lambda(f) = \int f g d\mu$$

but $g \in L_\infty(\Omega, \Sigma, \mu)$. L_∞ consists of functions g that are essentially bounded, i.e. there is a bound M such that $\mu[\omega : |g(\omega)| > M] = 0$. $\|g\|_\infty$ is the smallest M that works. $\|\Lambda\| = \|g\|_\infty$. Since

$$|\int f g d\mu| \leq \|f\|_p \|g\|_q$$

for conjugate pairs p, q the functions g in L_q do define bounded linear functionals with the correct bound. We concentrate now on the converse. Since μ is a finite measure, $\mathbf{1}_A(\omega) \in L_p$. Define

$$\lambda(A) = \Lambda(\mathbf{1}_A(\omega))$$

$$\|\lambda(A)| \leq C\|\mathbf{1}_A(\omega\|_p$$

$$\sup_{A\in\Sigma}|\lambda(A)| \le C\sup_{A\in\Sigma}\|\mathbf{1}_A(\omega\|_p = C[\mu(\Omega)]^{\frac{1}{p}}$$

To prove λ is a countably additive signed measure, we need to check that for a countable collection of pairwise disjoint sets A_i, with $\cup_{i=1}^\infty A_i = A$, we have

$$\|(\sum_{i=1}^n \mathbf{1}_{A_i}(\omega)) - \mathbf{1}_A(\omega)\|_p \to 0.$$

The difference is the indicator of the set $\cup_{i=n+1}^\infty A_i$ whose measure tends to 0 and so does its L_p norm for $1 \le p < \infty$. $\lambda(A)$ is a signed measure. $\lambda << \mu$. There is a Radon-Nikodym derivative.

$$\Lambda(\mathbf{1}_A) = \lambda(A) = \int_A g d\mu$$

with $g \in L_1$. $\Lambda(f) = \int f g d\mu$ for simple functions and then for bounded measurable functions. Take $f = (sign\ g)|g|^{q-1}\mathbf{1}_{|g|\le M}$. Then f is bounded and $pq = p + q$

$$\int |f|^p d\mu = \int_{|g|\le M} |g|^{pq-p} d\mu = \int_{|g|\le M} |g|^q d\mu$$

$$\Lambda(f) = \int_{|g|\le M} |g|^q d\mu \le C\Big[\int_{|g|\le M} |g|^q d\mu\Big]^{\frac{1}{p}}$$

$$\Big[\int_{|g|\le M} |g|^q d\mu\Big]^{\frac{1}{q}} \le C$$

Let $M \to \infty$. $g \in L_q$ and $\|g\|_q \le C$

If $p = 1$, $|\lambda(A)| \le C\mu(A)$. $g = \frac{d\lambda}{d\mu}$. $|g| \le C$ a.e. or $\|g\|_\infty \le C$.

ℓ_p **spaces.** The space of sequences $\xi = \{a_n\} : n \ge 1$.

$$\|\xi\|_p = \Big[\sum_{i=1}^\infty |a_n|^p\Big]^{\frac{1}{p}}$$

The Dual of ℓ_p is ℓ_q. $pq = p + q$. $\|\xi\|_\infty = \sup_n |a_n|$.