Weak Topology. A weak open set around $x \in \mathcal{X}$ is given by

$$N(x : n, \Lambda_1, \ldots, \Lambda_n) = \{y : |\Lambda_i(x) - \Lambda_i(y)| \le \epsilon, \forall\ 1 \le i \le n\}$$

for a finite collection of linear functionals $\{\Lambda_i\}$ in the dual $\mathcal{X}^*$ of $\mathcal{X}$. It is not metrizable! There is no countable basis at 0 unless $\mathcal{X}^*$ and therefore $\mathcal{X}$ is finite dimensional. But if $\mathcal{X}^*$ is separable then the unit ball, with weak topology is metrizable and is in fact compact. With a countable dense subset $\{\Lambda_i\}$ of $\mathcal{X}^*$

$$d(x, y) = \sum_{i=1}^{\infty} 2^{-i} |\Lambda_i(x) - \Lambda_i(y)|$$

will do it. We can try the weak topology on the dual $\mathcal{X}^*$. Either we can try the linear functionals $< \Lambda, x >= \Lambda(x)$ as linear in x for fixed Λ or linear in Λ for fixed x. So $\mathcal{X}^*$ has two weak topologies using linear functionals $x(\Lambda)$ from $\mathcal{X}$ or $x^{**}(\Lambda)$ from $\mathcal{X}^{**}$. Since $\mathcal{X} \subset \mathcal{X}^{**}$ one is weaker than the other. The weak topology on $\mathcal{X}^*$ can come from considering either $\mathcal{X}$ or $\mathcal{X}^{**}$. One hardly ever chooses $\mathcal{X}^{**}$. In many examples like L_p spaces with $1 < p < \infty$, $\mathcal{X} = \mathcal{X}^{**}$. Such spaces are called reflexive Banach spaces.

Weak compactness. The Unit Ball in L_p for $1 < p < \infty$ is compact in the weak topology.

L_1 is different. We have functions $f_n(x)$ such that $\int |f_n(x)| d\mu \le 1$ May not have a weak limit. For example $f_n(x) = n \mathbf{1}_{[0, \frac{1}{n}]}$ in $L_1[0, 1]$ with Lebesgue measure. The weak limit wants to be the δ-function at 0. Need uniform integrability.

A finite dimensional subspace of a Banach space is closed. Let $S = \{a_1 x_1 + \cdots + a_d x_d\}$ for some fixed lineraly independent $x_1, \ldots, x_d \in \mathcal{X}$ and $a_1, \ldots, a_d \in R^d$. Let $S \ni x_n = a_1^n x_1 + \cdots + a_d^n x_d$ and $x_n \to x$. If $\tau_n = \sup_{n,j} |a_j^n|$ is bounded then we can choose subsequences so that $a_j^n \to a_j$ and $x = a_1 x_1 + \cdots + a_d x_d \in S$. If τ_n is unbounded we can divide both sides of

$$x_n = a_1^n x_1 + \cdots + a_d^n x_d$$

by τ_n. The left side will $\to 0$. The terms on the right $\frac{\{a_j^n}{\tau_n\}}$ will be bounded and if we take a limit of subsequence $a_j^n \to a_j$ and at least one a_j will be such that $|a_j| = 1$.

$$\sum a_j x_j = 0$$

contradicting linear independence.

The unit ball $\|x\| \le 1$ can not be compact if $\mathcal{X}$ is not finite dimensional. Let $\mathcal{X}$ be infinite dimensional. Given any $\alpha < 1$ there is a sequence x_n such that $\|x_n\| = 1$ for all n and $\|x_i - x_j\| \ge \alpha$ for all $i \ne j$. It is enough to show that given a closed subspace $S \subset \mathcal{X}$, $S \ne \mathcal{X}$, and $\alpha < 1$, there is a $y \in \mathcal{X}$ such that $\|y\| = 1$ and $\inf_{x \in S} \|y - x\| \ge \alpha$.

Take $y \notin S$ with $\|y\| = 1$. Since S is closed $\inf_{x \in S} \|y - x\| = \theta > 0$ For any $\epsilon > 0$ can find $x_1 \in S$ such that $\|y - x_1\| \le \theta + \epsilon$. Let $y_1 = \frac{(y - x_1}{\|y - x_1\|}$. Then $\|y_1\| = 1$. Since S is a subspace for ϵ small

$$d(y_1, S) = d(\frac{y}{\|y - x_1\|}, S) = \frac{1}{\|y - x_1\|} d(y, S) \ge \frac{\theta}{\theta + \epsilon} \ge \alpha$$

Linear Operators. Compact Operators. Composition. Uniform Limits.

An operator T from $\mathcal{X}$ to $\mathcal{Y}$ is compact or completely continuous if the image of the unit ball of $\mathcal{X}$ is a compact set in$\mathcal{Y}$. T_1, T_2 compact implies $T_1 + T_2$ is compact. $T_1 : \mathcal{X} \to \mathcal{Y}$ $T_2 : \mathcal{Y} \to \mathcal{Z}$. If one of them is bounded and the other is compac the composition is compact. A bounded operator maps compact sets into compact sets.

T_n compact for each n,$\|T_n - T\| \to 0$ implies T is compact. Let $x_k \in \mathcal{X}$ satisfy $\|x_k\| \leq 1$. Since T_n is compact there is a subsequence such that $T_n x_k \to y_n$ as $k \to \infty$. We can diagonalize and assume this happens for all n. We want to show that Tx_k has a limit.

$$\|Tx_i - Tx_j\| \leq \|T_n x_i - T_n x_j\| + \|T_n - T\|\|x_i - x_j\|$$

$$\limsup_{i,j\to\infty} \|Tx_i - Tx_j\| \leq \|T_n - T\|\|x_i - x_j\| \leq 2\|T_n - T\|$$

Let $n \to \infty$.

Examples of compact operators.

1. $\mathcal{X} = C[0,1]$. $(Tf)(s) = \int_0^1 K(s,t)f(t)dt$ for a nice continuous function function K of two variables.

2. Let $x_1, x_2, \ldots, x_n \in \mathcal{X}$, $\Lambda_1, \ldots, \Lambda_n \in \mathcal{X}^*$. $Tx = \sum_{i=1}^n \Lambda_i(x)x_i$.

The adjoint. If $T : \mathcal{X} \to \mathcal{Y}$, $A^* : \mathcal{Y}^a st \to \mathcal{X}^*$ is defined by

$$< T^* y^*, x > = < y^*, Tx >$$

T bounded implies T^* is bounded by the same bound.

$$\|T\| = \sup_{|\|x\| \leq 1} \|Tx\| = \sup_{\substack{\|x\| \leq 1 \\ \|y^*\| \leq 1}} | < Tx, y > | = \sup_{\substack{\|x\| \leq 1 \\ \|y^*\| \leq 1}} | < x, T^* y > | = \sup_{\|y^*\| \leq 1} \|T^* y\| = \|T^*\|$$

If T is compact so is T^*. Let $K = T^* B_1$ the image of the unit ball. For any $\epsilon > 0$ we need to cover K by a finite number balls of radius ϵ. We can view $K \subset \mathcal{X}^*$ as functions on $\mathcal{X}$. If x_1^*, x_2^* are two members of K, $\|x_1^* - x_2^*\| = \|T^* y_1^* - T^* y_2^*\|$ for some $y_1^*, y_2^* \in B_1(\mathcal{Y}^*)$.

$$\begin{aligned}
\|T^* y_1^* - T^* y_2^*\| &= \sup_{\|x\| \leq 1} | < T^*(y_1^* - y_2^*), x > | \\
&= \sup_{\|x\| \leq 1} |(y_1^* - y_2^*), Tx > | \\
&= \sup_{y \in TB_1(\mathcal{X})} | < y_1^* - y_2^*, y > |
\end{aligned}$$

The linear functionals $< y^*, y >$ are continuous on the compact set $K_1 = TB_1(\mathcal{X}$ and satisfy a uniform estimate $| < y^*, y_1 - y_2 > | \leq \|y_1 - y_2\|$. They are uniformly bounded. By Ascoli-Arzela theorem the space of functions is compact and can be covered by a finite number of balls.

Hilbert Spaces. A Hilbert space $\mathcal{H}$ is a vector space with an inner product $\langle x, y\rangle$ that satisfies

1. $\langle x, y\rangle = \langle y, x\rangle$ is linear in x for each y and linear in y for each x.

2. $\langle x, x\rangle > 0$ for $x \neq 0$.

It follows that

$$\langle (y+tx), (y+tx)\rangle = \langle y, y\rangle + 2t\langle x, y\rangle + t^2\langle x, x\rangle \geq 0$$

and

$$\langle x, y\rangle^2 \leq \langle x, x\rangle\langle y, y\rangle$$

and if we define $\|x\| = \sqrt{\langle x, x\rangle}$ then $|\langle x, y\rangle| \leq \|x\|\|y\|$ and $\|x\|$ is a norm on $\mathcal{H}$.

3. The space $\mathcal{H}$ is complete under the norm $\|x\|$.

Two vectors x_1, x_2 are orthogonal if $\langle x_1, x_2\rangle = 0$. Denoted by $x_1 \perp x_2$.

A collection $\{x_\alpha\}$ is orthonormal if $x_\alpha \perp x_\beta$ for $\alpha \neq \beta$ and $\|x_\alpha\| = 1$ for all α.

A complete orthonormal set is a maximal orthonormal collection $\{x_\alpha\}$ such that if $x \perp x_\alpha$ for α then $x = 0$.

We will assume that our Hilbert Space $\mathcal{H}$ is separable. Since $\|x_\alpha - x_\beta\| = \sqrt{2}$ if $\alpha \neq \beta$ in an orthonormal set, any orthonormal set in a separable space has to be countable.

Given any set of n mutually orthogonal vectors $x_1, x_2, \ldots, x_n \in \mathcal{H}$, and a additional vector y linearly independent of $x_1, x_2, \ldots, x_n$, there exists $x_{n+1} = c_{n+1}[y - \sum_{j=1}^n c_j x_j]$ such that $x_1, x_2, \ldots, x_n, x_{n+1}$ is a set of $n+1$ orthonormal vectors and span the same subspace as $x_1, x_2, \ldots, x_n, y$. For $1 \leq j \leq n$, $< x_{n+1}, x_j >= 0$ yields $< y, x_j >= c_j$ We need to determine c_{n+1}. To this end

$$< x_{n+1}, x_{n+1} >= c_{n+1}^2 \left[\|y - \sum_{j=1}^n c_j x_j\|^2\right] = 1$$

Finally need to check that $\|y\|^2 > \sum_{j=1}^n c_j^2$. Since y is not in the span of $x_1, \cdots, x_n$ $\|y - \sum_{j=1}^n c_j x_j\| > 0$. It follows that any separable Hilbert space has a countable orthonormal set that spans $\mathcal{H}$, i.e an orthonormal basis. Start with a countable dense set and trim it to a linearly independent set that spans $\mathcal{H}$ and then replace them inductively by an orthonormal set. This is known as the Gram-Schmidt process.You end with an orthonormal basis. Complete Orthonormal Set. $\{x_j\}$. If $y \perp x_j$ for all j then $y = 0$.

$\{e_i\}$ is an orthonormal set of vectors. The following are equivalent

1. $\{e_i\}$ is maximal. That is if $x \perp e_i$ for all i then $x = 0$

2. For any $y \in \mathcal{H}$, $\|y\|^2 = \sum_i < y, e_i >^2$

3. For any $y \in \mathcal{H}$, $y = \sum_i < y, e_i > e_i$

Proof. $3 \Rightarrow 2 \Rightarrow 1$ is obvious. Need to prove $1 \Rightarrow 3$

$$\|y\|^2 \geq \sum_i < y, e_i >^2$$

$$< y - \sum_i < y, e_i > e_i, e_j >= 0$$

for all j. Therefore $y - \sum_i < y, e_i > e_i = 0$ because of maximality.

The space l_2. Sequences $x = \{a_1, a_2, \ldots\}$ that are square summable, i.e $\sum_{j=1}^{\infty} a_j^2 < \infty$. $< x, y >= \sum_{j=1}^{\infty} a_j b_j$

Weak Convergence. $< x_n, y > \to < x, y >$ for all $y \in \mathcal{H}$

If x_n converges weakly then $\|x_n\|$ is bounded. An application of Baire Category Theorem.

$$\mathcal{H} = \cup_k \{y : \sup_n | < x_n, y > | \leq k\}$$

For some k, $\{y : \sup_n | < x_n, y > | \leq k\}$ has interior. In other words for some x_0, k and δ

$$\sup_{\|y - x_0\| < \delta} \sup_n | < x_n, y > | \leq k$$

or

$$\sup_{\|y\| < 1} \sup_n | < x_n, y > | \leq \frac{2k}{\delta}$$

.

Unit Ball is weakly compact. $< x, y >$ is jointly continuous in the strong or norm topology. $< x_n, y_n > \to < x, y >$ if either $x_n \to x$ strongly or $y_n \to y$ strongly while the other can converge weakly. If both converge weakly it may not converge. In fact if $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ then $\|x_n - x\| \to 0$.

There is only one Hilbert Space of given dimension. Finite dimension d. Countable infinite dimension. Any correspondence between complete orthonormal basis sets up an isomorphism. In particular $\mathcal{H}^* = \mathcal{H}$. The adjoint T^*x is defined by $< T^*x, y >=< x, Ty >$ for all y. Self adjoint operators are those for which $T^* = T$, or $< Tx, y >=< x, T^*y > \ \forall\ x, y$.

Eigen Values, Eigen functions etc. May not exist. Compact Self adjoint operators have a complete orthonormal set of eigen functions, with eigenvalues accumulating at 0.