1. Consider the space $\mathbf{Z} = \{1, 2, 3, \ldots\}$ of positive integers. $d(i, j) = |2^{-i} - 2^{-j}|$. Is $(\mathbf{Z}, d)$ a metric space? Is it complete, Is it compact, is it separable?

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3. Consider the space $X = [0, 1]$ the interval $0 \leq x \leq 1$ and the sequence of functions $f_n(x) = nx^n$ defined for $n \geq 1$ on X. Let $f(x)$ be the function $f(x) = 0$ for all x.

Does $f_n \to f$ uniformly on $[0, 1]$. Does it converge at every point $x \in X$? Does it converge almost everywhere with respect to Lebesgue measure? Is the sequence dominated by an integrable function, is the family uniformly integrable. Does $\int_0^1 f_n(x)dx \to 0$?

4. If $K(x, y)$ is a continuous symmetric function on $[0, 1] \times [0, 1]$ show that operator is compact. Let $\{\lambda_j\}$ be the eigenvalues and $\{f_j(x)\}$ the complete orthonormal set of the corresponding eigenfunctions. Show that

$$K(x, y) = \sum_j \lambda_j f_j(x) f_j(y)$$

and the convergence takes place in $L_2\big[[0, 1] \times [0, 1]\big]$. In paticular

$$\sum_j |\lambda_j|^2 = \int_0^1 \int_0^1 |K(x, y)|^2 dx dy < \infty$$