Problemset 1. Due Feb 28.

1. Let P and Q be two probability measures on (X, Σ). Let $\mathcal{P}_n$ be an increasing family of measurable finite partitions of X, i.e $\mathcal{P}_n = \{A_1, \ldots, A_n\}$ with $X = \cup_{i=1}^n A_j$ with disjoint A_j's from Σ and $\mathcal{P}_{n+1}$ is a finer partition than $\mathcal{P}_n$. The Radon Nikodym derivative on $\mathcal{P}_n$ is defined by

$$\frac{dQ}{dP}\Big|_{\mathcal{P}_n} = f_n(x) = \sum \frac{Q(A_j)}{P(A_j)} \mathbf{1}_{A_j}(x)$$

Show that $f_n(x)$ has an almost sure limit with respect P and Q. If we decompose X as $X_1 \cup X_2 \cup X_3$, where $Q(X_1) = 0$, $P(X_3) = 0$ and P and Q are mutually absolutely continuous on X_2, then the limit f of f_n satisfies $f = \infty$ on X_3, $f = 0$ on X_1 and $0 < f < \infty$ on X_2

2. $P_{i,j}$ are the transition probabilities of a Markov Chain on the set of positive integers $\mathbf{Z}^+ = 1, 2, \ldots, n, \ldots$ such that $P_{i,i\pm1} > 0$ and $P_{i,j} = 0$ unless $|i - j| \leq 1$. If there is a positive function $U(k)$ such that $U(k) \to 0$ as $k \to \infty$ and

$$P_{i,i+1}U(i+1) + P_{i,i}U(i) + P_{i,i-1}U(i-1) \leq U(i)$$

for i sufficienly large, then show that the Markov Chain is transient. If for large i, $P_{i,i} = 0$ and $P_{i.i+1} \geq \frac{1}{2} + \frac{a}{i}$ for some $a > \frac{1}{4}$ then show that such a U exists. What can you say if $P_{i.i+1} = \frac{1}{2} + \frac{1}{4i}$?

3. Let Ω be the circle $|z| = 1$ and μ be the normalized arc length. Consider $T : z \to e^{i\alpha} z$ for some real α. When is it measure preserving? When is it ergodic? Provide proof for ergodicity.

4. Let (Ω, Σ, P) be a probability space and the sub σ-fields be $\downarrow$, i.e. $\mathcal{F}_n \supset \mathcal{F}_{n+1}$ for all $n \geq 1$. Assume that we have a (reverse) martingale

$$f_{n+1} = E[f_n | \mathcal{F}_{n+1}]$$

Then for $p \geq 1$, if $f_1 \in L_p(\Omega)$, then so does f_n for all n and $f_n \to f_\infty$ a.e and in $L_p(\Omega)$ where

$$f_\infty = E[f_1 | \mathcal{F}_\infty]$$

with $\mathcal{F}_\infty = \cap_n \mathcal{F}_n$