$\{X_n\}$ is a Markov Chain on the integers $i = 0, 1, \ldots$ with transition probabilities

$$p(i,j) = e^{-i} \frac{i^j}{j!}$$

This is an example of a branching process, where each member of the current generation has a random number of offsprings distributed according to a Poisson distribution with parameter 1, and the number of offsprings is independent for different members. Then the population size at generation $k+1$ is distributed according to a Poisson with parameter $i$ if the size of the population is $i$ in the $k$-th generation.

1) Show that the population eventually dies out with probability 1. i.e

$$P[X_n = 0 \text{ for some } \ n] = 1.$$

Of course if $X_n = 0$ then $X_m = 0$ for $m \geq n$.

Let us start with a large population of size $Nx$. Consider the population size $X_{Nt}$ at time $Nt$ and define $x_N(t) = \frac{X_{Nt}}{N}$.

2) Show that as $N \to \infty$ there is a limiting process $x(t)$ of the "size" which is the diffusion with generator

$$\frac{x}{2} \frac{d^2}{dx^2}$$

starting from $x(0) = x$.

Consider the evolution of $k$ such populations independently of each other with generator

$$\frac{1}{2} \sum_{i=1}^{k} x_i \frac{\partial^2}{\partial x_i^2}$$

3) Show that $S(t) = \sum_{i=1}^{k} x_i(t)$ is a Markov process and find its generator.

4) Show that $y(t) = \{y_i(t) : 1 \leq i \leq k\}$ where $y_i(t) = \frac{x_i(t)}{S(t)}$ lives on the simplex

$$D = \{y : y_i \geq 0, \sum_i y_i = 1\}$$

is Markov and find its generator.

5) Show that the process $y(t) = \{y_i(t)\}$ moves successively through faces, one dimension lower each time until it reaches some vertex $P_i$ with $y_i = 1$ and $y_j = 0$ for $j \neq i$ and then stays there for ever.

6) Find the probability $u_i(y)$ that the process is absorbed at the vertex $P_i$, if it starts from $y = (y_1, \ldots, y_k)$.

7) If $\tau$ is the time of absorption at a vertex calculate $m(y) = E_y[\tau]$.

**Hint.** The hardest part perhaps is to show that the process does not lose two dimensions at the same time. i.e hits an 'edge' rather than a face. This amounts to proving that with $k = 2$ the process corresponding to

$$\frac{1}{2}\sum_{i=1}^{2} x_i \frac{\partial^2}{\partial x_i^2}$$

does not exit from $x_1 > 0, x_2 > 0$ at $(0,0)$. If $\tau_i$ is the exit time $\tau_i = \inf\{t : x_i(t) = 0\}$ we have to show $P_{x_1,x_2}\{\tau_1 = \tau_2\} = 0$. Since $\tau_1$ and $\tau_2$ are independent this amounts to showing that $\tau$, the hitting time of 0 for the one dimensional process

$$\frac{x}{2}\frac{d^2}{dx^2}$$

has a continuous distribution function. There is a trick for proving it. Let $\tau_a$ be the hitting time of $a$ and for $b > a$ let

$$f(\lambda, b, a) = E_b[e^{-\lambda\tau_a}]$$

Then for $c > b > a$ by the strong Markov property $f(\lambda, c, a) = f(\lambda, c, b)f(\lambda, b, a)$. More over $f(\lambda, x, a)$ is the solution of

$$\frac{x}{2}\frac{d^2 f}{dx^2} = \lambda f$$

with $f(\lambda, a, a) = 1$. Therefore $f(\lambda, b, a) = f(1, \lambda b, \lambda a)$. In particular

$$f(\lambda, x, 0) = f(\lambda, x, \frac{x}{2})f(\frac{\lambda}{2}, x, 0)$$

The distribution function $F_x(t)$ of $\tau_0$ under $P_x$ satisfies

$$F_x(t) = F_{\frac{x}{2}}(t) * G_x(t) = F_x(2t) * G_x(t)$$

where $G_x$ is the distribution function of $\tau_{\frac{x}{2}}$ under $P_x$. This shows that the biggest jump $j(x)$ in the distribution function $F_x(t)$ has to be 0. An alternate method is to construct a barrier, i.e a function $U(x_1, x_2) \geq 0$ satisfying

$$\frac{1}{2}\sum_{i=1}^{2} x_i \frac{\partial^2 U}{\partial x_i^2} = 0$$

which blows up near $(0,0)$. Then the process cannot approach $(0,0)$. Such a function will not be smooth as $x_1$ or $x_2$ tend to 0. One can construct such a $U$ by separation of variables by trying

$$U(x_1, x_2) = (x_1 + x_2)^{-\alpha} f(\frac{x}{x+y})$$

and solving an ODE for $f$. This is a model for several non competing species, with no advantage for any and they disappear due to chance, one species at a time, until only one survives. $x(t)$ describes the total sizes and $y(t)$ the proportions.