1. P and Q are absolutely continuous with respect to $P+Q$. $g_n = \frac{dP}{d(P+Q)}\Big|_{\mathcal{P}_n} = \frac{f_n}{f_n+1}$ has a limit $g = \frac{dP}{dQ}\Big|_{\Sigma}$ as $n \to \infty$. $0 \le g \le 1$. $g = 0$ on X_3 and $g = 1$ on X_1 with $0 < g < 1$ on X_2. If $g_n = \frac{f_n}{f_n+1}$ then $f_n = \frac{g_n}{1-g_n}$ tends to $f = \frac{g}{1-g}$. $g = 1$ on X_1 so $f = \infty$.

2. Let is take a large k_0 so that the inequality

$$P_{i,i+1}U(i+1) + P_{i,i}U(i) + P_{i,i-1}U(i-1) \le U(i)$$

is valid for $i \ge k_0$. Let τ_{N,k_0} be the hitting time of $[k_0, N]$. The for $k > k_0$, $u(X_n)$ is a super-martingale. Therefore

$$P_k[X_{\tau_{N,k_0}} = k_0]U(k_0) + P_k[\tau_{N,k_0} = N]U(N) \le U(k)$$

In particular

$$P_k[X_{\tau_{N,k_0}} = k_0]U(k_0) \le U(k)$$

Letting $N \to \infty$,

$$P_k[\tau_{k_0} < \infty] \le \frac{U(k)}{U(k_0)}$$

Since $U(k) \to 0$, this proves transience.

Let us try and solve

$$U(i+1)(\frac{1}{2} + \frac{a}{i}) + U(i-1)(\frac{1}{2} - \frac{a}{i}) = U(i)$$

with $v(i) = U(i+1) - U(i)$, we have

$$v(i)(\frac{1}{2} + \frac{a}{i}) = V(i-1)(\frac{1}{2} - \frac{a}{i})$$

or

$$\frac{v(i)}{v(i-1)} = \frac{(\frac{1}{2} - \frac{a}{i})}{(\frac{1}{2} + \frac{a}{i})} = \frac{(1 - \frac{2a}{i})}{(1 + \frac{2a}{i})} \simeq (1 - \frac{4a}{i})$$

This gives

$$v(i) \simeq \frac{1}{i^{4a}}$$

If $4a > 1$, then $\sum_i v(i) < \infty$ and one can fix it so that $U(i) \to 0$ as $i \to \infty$. $U(k) = \sum i = k^{\infty} v(i)$ will do it. If $4a = 1$, then $U(k) \simeq \log k$ and this will give recurrence.

3. No matter what α is this is rotation and preserves arc length. If α is rational it is NOT ergodic. If α is irrational then, the proof of ergodicity depends on showing for $f \in L_2(\mu)$

$$\frac{1}{n}\sum_{j=1}^{n} f(T^j z) \to \int f d\mu$$

where μ is the normalized arc length. Trivial if $f = 1$. Enough to show for a dense set. Therefore enough to show for $f = e^{ik\theta}$. In that case it reduces to

$$\frac{1}{n}\sum_{j=1}^{n} e^{ikj\alpha} \to 0$$

which is true if $e^{ik\alpha} \neq 1$ for $k \neq 0$, i.e. α is irrational.

4. The proof proceeds just like the usual case. First prove it for $f \in L_2(P)$. If $\mathcal{H}_n$ are decreasing subspaces in a Hilbert space with $\mathcal{H}_n \downarrow \mathcal{H}_\infty$, then the projections π_n converge to π_∞. This is converted to the increasing case by orthogonal complementation. Since L_2 is dense in every L_p for $1 \leq p < \infty$, and the conditional expectation is a contraction in every L_p including $p = 1$, every (reverse) martingale converges in L_p if we start with $f \in L_p$ so long as $1 \leq p < \infty$. The almost sure convergence is a consequence of Doob's inequality that will givea bound and the upcrossing inequality which will control the oscillations.