2. Independent random variables.

The Law of large Numbers. If $\{X_i : i \geq 1\}$ are a sequence of independent identically distributed random variables (on some $(\Omega, \mathcal{F}, P)$) with $E[X_i] = m$, then

$$P\big[\omega : \lim_{n\to\infty} \frac{1}{n}[X_1(\omega) + \cdots + X_n(\omega)] = m\big] = 1$$

This is the strong law of large numbers. The weak law, which is naturally weaker than the strong law, asserts that for any $\epsilon > 0$,

$$\lim_{n\to\infty} P\big[\omega : \big|\frac{1}{n}[X_1(\omega) + \cdots + X_n(\omega)] - m\big| \geq \epsilon\big] = 0$$

The strong law requires the existence of the mean, i.e $\{X_i\}$ have to be integrable. On the other hand the weak law can be valid some times even if we can only have the mean defined as

$$m = \lim_{a\to\infty} \int_{-a}^{a} x \, dF(x)$$

where F is the common distribution of $\{X_i\}$.

The Central Limit Theorem. If $\{X_i\}$ are independent, identically distributed and have mean 0 and variance σ^2, then the distribution of $Z_n = \frac{1}{\sqrt{n}}[X_1 + \cdots + X_n]$, approaches the normal distribution with mean , i.e.

$$\lim_{n\to\infty} P[\frac{1}{\sqrt{n}}[X_1 + \cdots + X_n] \leq x] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{y^2}{2\sigma^2}} dy$$

The central limit theorem is valid also if $\{X_i\}$ are not identically distributed, but X_i has mean 0 and variance σ_i^2. Then with $Z_n = \frac{1}{s_n}[X_1 + \cdots + X_n]$, where $s_n^2 = \sum_{i=1}^{n} \sigma_i^2$, we have

$$\lim_{n\to\infty} P[Z_n \leq x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$$

provided $s_n \to \infty$ and some additional condition known as Lindeberg's condition is satisfied. It is that, for any $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{i=1}^{n} \int_{|x|\geq \epsilon s_n} x^2 dF_i(x) = 0$$

where F_i is the distribution of X_i.

Kolmogorov's Two Series Theorem. Let $\{X_i\}$ be a sequence of independent random variables uniformly bounded by a constant C with X_i having mean 0 and variance σ_i^2 . Then the necessary and sufficient condition for the convergence with probability 1 of the series

$$S = \sum_{i=1}^{\infty} X_i$$

is

$$\sum_{i=1}^{\infty} \sigma_i^2 < \infty$$

The sufficiency part of the proof depends on the important inequality known as Kolmogorov's inequality.

Kolmogorov's Inequality. Let $\{X_i\}$ be indpendent random variables with mean 0 and variance σ_i^2. Then

$$P[\sup_{1 \le j \le n} |X_1 + X_2 + \cdots + X_j| \ge \ell] \le \frac{1}{\ell^2} \sum_{I=1}^{n} \sigma_i^2$$

Tchebecheff's inequality gives

$$P[|X_1 + X_2 + \cdots + X_j| \ge \ell] \le \frac{1}{\ell^2} \sum_{I=1}^{j} \sigma_i^2$$

Naive calculation will yield

$$P[\sup_{1 \le j \le n} |X_1 + X_2 + \cdots + X_j| \ge \ell] \le \frac{1}{\ell^2} \sum_{j=1}^{n} \sum_{I=1}^{j} \sigma_i^2$$

which is not good enough. Let us define

$$E_j = \{\omega : |X_1 + X_2 + \cdots + X_k| < \ell \quad \text{for } k \le (j-1), \quad |X_1 + X_2 + \cdots + X_j| \ge \ell\}$$

We are interested in estimating

$$P(E) = \sum_{j=1}^{n} P(E_j)$$

Denoting by $S_j = \sum_{i=1}^{j} X_i$,

$$\begin{aligned}
\int_{E_j} S_n^2 dP &= \int_{E_j} (S_j + (S_n - S_j))^2 dP \\
&= \int_{E_j} [S_j^2 + 2S_j(S_n - S_j) + (S_n - S_j)^2] \, dP \\
&\ge \int_{E_j} S_j^2 \, dP \ge \ell^2 P(E_j)
\end{aligned}$$

The cross term is 0 because E_j, S_j are independent of $S_n - S_j$. Because E_j are disjoint and their union is E, summing over $j = 1, 2, \ldots, n$, we get

$$s_n^2 \ge \int_E s_n^2 dP \ge \ell^2 P(E)$$

This allows the tail of the series

$$T_n = \sup_{j \geq n} |S_j - S_n|$$

to be estimated by

$$P[T_n \geq \epsilon] \leq \frac{1}{\epsilon^2} \sum_{j=n+1}^{\infty} \sigma_j^2$$

proving $T_n \to 0$ with probability 1. The converse depends on an inequality as well. Assume $\limsup |S_n| < \infty$ with positive probbaility. Let

$$F_n = \{\omega : \sup_{1 \leq j \leq n} |S_j| \leq \ell\}$$

so that $E_{n+1} = F_n - F_{n+1}$ are disjoint. Then there is an ℓ such that $P(F_n) \geq \delta > 0$ for all

$$\begin{aligned}
\int_{F_{n+1}} S_{n+1}^2 dP + \int_{E_{n+1}} S_{n+1}^2 dP - \int_{F_n} S_n^2 dP &= \int_{F_n} S_{n+1}^2 dP - \int_{F_n} S_n^2 dP \\
&= \int_{F_n} [2X_{n+1}S_n + X_{n+1}^2] dP \\
&\geq \int_{F_n} X_{n+1}^2 \, dP \\
&= P(F_n)\sigma_{n+1}^2 \\
&\geq \delta \, \sigma_{n+1}^2
\end{aligned}$$

On the other hand $|S_{n+1}| \leq C + \ell$ on E_{n+1}. Therefore

$$\sigma_{n+1}^2 \leq \frac{1}{\delta}[(C+\ell)^2 P(E_{n+1}) + \int_{E_{n+1}} S_{n+1}^2 dP - \int_{F_n} S_n^2 dP]$$

And the telescoping sum

$$\sum [\int_{E_{n+1}} S_{n+1}^2 dP - \int_{F_n} S_n^2 dP]$$

is bounded by ℓ^2. Providing the estimate

$$\sum_j \sigma_j^2 \leq \frac{1}{\delta}[(C+\ell)^2 + \ell^2]$$

Actually we have shown some thing stronger. If

$$\sum_j \sigma_j^2 = +\infty$$

then

$$P[\limsup_{n\to\infty} |S_n| = \infty] = 1$$

An easy corollary is that if $\{X_i\}$ are independent random variables with $E[X_i] = 0$ and $E[X_i^2] = \sigma_i^2$ then for any sequence of constants $\{a_j\}$ satisfying

$$\sum_j \sigma_j^2 a_j^2 < \infty$$

the series

$$S = \sum_j a_j X_j$$

will converge with probability 1,