5. Markov Processes.

A stochastic process in discrete time is just a sequence $\{X_j : j \geq 0\}$ of random variables with values in some $(\mathcal{X}, \mathcal{F})$ defined on a probability (Ω, Σ, P). It can also be specified by prescribing, in a self consistent manner, the joint distribution of $\{X_0, X_1, X_2, \ldots, X_n\}$ for every n. A convenient way of doing it is by specifying the the distribution $p_0(dx_0)$ of X_0 and the conditional distributions

$$p_n(x_0, x_1, \ldots, x_{n-1}; dx_n)$$

of X_n given $X_0, \ldots, X_{n-1}$. Ω can be the product space $\mathcal{X}^\infty$, i.e. the space of sequences with values in $\mathcal{X}$. There is a canonical P on the natural σ-field $\mathcal{F}_\infty$ on Ω. There is also the sub-σ-fields $\mathcal{F}_n$ generated by $x_0, x_1, \ldots, x_n$. The canonical P will equal p_0 on $\mathcal{F}_0$ and the conditional distribution of on $\mathcal{F}_n$ given $\mathcal{F}_{n-1}$ will be given by $p_n(x_0, x_1, \ldots, x_{n-1}; dx_n)$. In the special case when $p_n(x_0, x_1, \ldots, x_{n-1}; dx_n) = \pi_n(x_{n-1}, dx_n)$, for $n \geq 1$, depends only on x_{n-1}, the process is called a Markov Process. Of course when they are just $p_n(dx_n)$ and do not depend on any x_i for $0 \leq i \leq n-1$ we have independent random variables and P is the product measure. If, in the Markov case, $\pi_n(\cdot, \cdot)$ is the same $\pi(\cdot, \cdot)$ for all $n \geq 1$, it is called a Markov process with stationary transition probabilities.

A simple example is to take $\mathcal{X}$ to be a countable set. Then p_0 is just the set of probabilities $p_0(x) = P[X_0 = x]$, and

$$P[X_n = y | X_0 = x_0, \cdots, X_{n-1} = x_{n-1}] = \pi_n(x, y)$$

are the transition probabilities which in the stationary case is independent of n. It is natural to consider $(\Omega, \mathcal{F}_n, \mathcal{F}_\infty, P)$. There are some natural martingales. For simplicity we limit ourselves to the stationary case.

Theorem. For any function f on $\mathcal{X}$ let us define

$$(\pi f)(x) = \sum_y \pi(x, y) f(y) = E[f(X_n) | X_{n-1} = x]$$

Then

$$Z_n = f(X_n) - f(X_0) - \sum_{j=0}^{n-1} (\pi f - f)(X_j)$$

is a martingale with respect to $(\Omega, \mathcal{F}_n, P)$.

Proof: Let us compute $E[Z_n|\mathcal{F}_{n-1}]$.

$$\begin{aligned}
E[Z_n|\mathcal{F}_{n-1}] &= E[f(X_n)|\mathcal{F}_{n-1}] - \sum_{j=0}^{n-1}(\pi f - f)(X_j) \\
&= (\pi f)(X_{n-1}) - \sum_{j=0}^{n-1}(\pi f - f)(X_j) \\
&= f(X_{n-1}) - \sum_{j=0}^{n-2}(\pi f - f)(X_j) \\
&= Z_{n-1}
\end{aligned}$$

Remark. If we replace the definition $(\pi f)(x) = \sum_y \pi(x,y)f(y)$ with

$$(\pi f)(x) = \int f(y)\pi(x, dy)$$

then the theorem is true for Markov processes on any state space. For simplicity we will assume that we have a countable state space.

Martingales are a useful tool in studying Markov Processes. Let us look at some examples.

1. Let $A \subset \mathcal{X}$. Define

$$\tau_A = \inf\{j : X_j \in A\}$$

is the first hitting time of A. It is possible that X_j never hits A in which case we take $\tau_A = \infty$. We wish to calculate for $\lambda > 0$,

$$\phi_\lambda(x) = E[e^{-\lambda \tau_A}|X_0 = x] \tag{5.1}$$

Then if $x \in A$ then $\phi_\lambda(x) = 1$. Moreover for $x \notin A$ it is easy to see that

$$\phi_\lambda(x) = e^{-\lambda}\sum_y \pi(x,y)\phi_\lambda(y)$$

Clearly $0 \leq \phi_\lambda(x) \leq 1$. We will show that the only bounded solution of

$$F(x) = e^{-\lambda} \sum_y \pi(x,y)F(y) \tag{5.2}$$

for $x \notin A$ with $F(x) = 1$ for $x \in A$ is given by (5.1). Let $F(x)$ be a solution of (5.2). Define

$$Z_n = e^{-\lambda\, n}\, F(X_n)$$

Then with $\Sigma_n = \sigma\{X_0, X_1, \ldots, X_n\}$,

$$\begin{aligned} E[Z_{n+1}|\Sigma_n] = e^{-\lambda\,(n+1)} E[F(X_{n+1}|\Sigma_n] \\ &= e^{-\lambda\,(n+1)} \sum_y \pi(X_n, y)F(y) \\ &= e^{-\lambda\, n} F(X_n) \\ &= Z_n \end{aligned}$$

provided $X_n \notin A$. One can rewrite this as

$$E[Z_{n+1} - Z_n|\sigma_n] = \begin{cases} 0 & \text{if } X_n \notin A \\ e^{-\lambda\, n} G(X_n) & \text{if } X_n \in A. \end{cases}$$

with

$$G(x) = e^{-\lambda} \sum_y \pi(X_n, y)F(y) - F(x)$$

Therefore

$$Z_n - Z_0 - \sum_{j=0}^{n-1} e^{-\lambda\, n} G(X_j) \mathbf{1}_A(X_j)$$

is a martingale. Let τ_A is a stopping time and for $n \leq \tau_A$ $X_n \notin A$ and $G(X_n) = 0$. Therefore $\{Z_n\}$ is bounded uniformly until τ_A even if τ_A itself can be large. Doob's stopping theorem applies and

$$E[e^{-\lambda\, \tau_A}] = E[e^{-\lambda\, \tau_A} F(X_{\tau_A})] = E[Z_\tau] = E[Z_0] = F(x)$$

Example. Consider the random walk on Z where $\pi(x, x \pm 1) = \frac{1}{2}$. If one starts from 0, and τ is the first time $\pm k$ is reached calculate $E[e^{-\lambda\, \tau}]$. Solve the equation

$$F(x) = e^{-\lambda} [\frac{1}{2} F(x-1) + \frac{1}{2} F(x+1)]$$

for $|x| \leq k-1$, with $F(x) = 1$ for $|x| \geq k$. One can isolate $[-k, k]$. Need to solve

$$F(x-1) + F(x+1) - 2e^{\lambda} F(x) = 0$$

with $F(\pm k) = 1$. Solve the quadratic

$$\rho^2 - 2e^{\lambda}\rho + 1 = 0$$

with roots

$$\rho_{\pm} = e^{\lambda} \pm \sqrt{e^{2\lambda} - 1} = e^{\pm\theta}$$

where $\theta = \log[e^{\lambda} + \sqrt{e^{2\lambda} - 1}]$. The solution is seen to be

$$F(x) = \frac{e^{\theta\, x} + e^{-\theta\, x}}{e^{\theta\, k} + e^{-\theta\, k}}$$

and

$$F(0) = [\cosh(\theta\, k)]^{-1}$$

Exercise. Start from $x > 0$. Show that sooner or later 0 is reached. Calculate $E[e^{-\lambda\,\tau}]$ where τ is the first time 0 is reached.

Exercise. What happens when

$$p = \pi(x, x-1) > \frac{1}{2} > \pi(x, x+1) = q = 1 - p$$

and when

$$p = \pi(x, x-1) < \frac{1}{2} < \pi(x, x+1) = q = 1 - p$$

Example. A game is being played where the probability is $\frac{1}{2}$ for each of two players to win any one round. It is agreed that the first person to win k rounds will be the winner. They put equal amounts to make a kitty for the winner to take. Unfortunately the game is interrupted before either player can win k rounds. It stops when player A needs to win a more rounds, and player B needs b rounds. $1 \leq a \leq k$, $1 \leq b \leq k$. What is the "fair" way to divide the kitty between the two players?

Let $u(a, b)$ be the proportion of the kitty that player A should get in a fair division when he needs a rounds and player B needs b rounds. Since the game is fair neither player can expect to gain or lose by playing an extra game.

$$u(a, b) = \frac{1}{2}u(a-1, b) + \frac{1}{2}u(a, b-1)$$

$u(0, b) = 1$ if $b > 0$ and $u(a, 0) = 0$ if $a > 0$. Solution is

$$u(a, b) = \frac{1}{2^{a+b-1}} \sum_{a+b-1 \geq r \geq a} \binom{a+b-1}{r}$$

You can verify that this is a solution. Can you show directly ?