Answers.

1.1. Clearly

$$f_n(X_1, X_2, \ldots, X_n) = \begin{cases} \frac{3}{4} & \text{if } (n+1)\text{-th toss uses coin 1} \\ \frac{1}{4} & \text{if } (n+1)\text{-th toss uses coin 2} \end{cases}$$

For the $n+1$-th coin to be coin 1, even number changes are needed. $n - S_n$ should be even. Therefore

$$f_n(X_1, X_2, \ldots, X_n) = \begin{cases} \frac{3}{4} & \text{if } n - S_n \text{ is even} \\ \frac{1}{4} & \text{if } n - S_n \text{ is odd} \end{cases}$$

1.2.

$$E[S_{n+1}^2 - S_n^2 | \mathcal{F}_n] = E[x_{n+p1}^2 + 2X_{n+1}S_n | \mathcal{F}_n] = 1$$

Hence $S_n^2 - n$ is a martingale. Let τ be the stopping time. Then for any k

$$E[S_{\tau\wedge k}^2 - \tau \wedge k] = x^2$$

or

$$E[S_{\tau\wedge k}^2] = E[\tau \wedge k] + x^2$$

S_n for $n \leq \tau$ is bounded by N. Hence we can let $k \to \infty$ in the LHS, and by the monotone convergence theorem it is OK to let $k \to \infty$ in the RHS. Therefore

$$E[S_\tau^2] = N^2 P[S_\tau = N] + 0^2 P[S_\tau = 0] = N^2 \frac{x}{N} = Nx = E[\tau] + x^2$$

or

$$E[\tau] = Nx - x^2 = x(N - x)$$

2.1 Let $P(x)$ be the probability that $\xi_n \to \infty$ given that $\xi_0 = x$. Then

$$P(x) = pP(x+1) + qP(x-1) \quad \text{for} \quad x \geq 1; \qquad P(0) = P(1)$$

This yields

$$(P(x+1) - P(x))p = q(P(x) - P(x-1))$$

Since $P(1) - P(0) = 0$ it follows that $P(x) \equiv c$. If we take

$$u(x) = A\rho^x$$

then this will solve

$$u(x) = pu(x+1) + qu(x-1)$$

if

$$\rho = p\rho^2 + q$$

or

$$\rho = \frac{1 + \sqrt{1-4pq}}{2p} = \frac{1 \pm (p-q)}{2p} = \{1, \frac{q}{p}\}$$

If we look at the entire set of integers and define $\pi(\ ,\ )$ as just a random walk then $u(\xi_n)$ will be a martingale. If τ is the time of hitting 0, there is no difference between the two. Hence

$$u(x) = \left(\frac{q}{p}\right)^x$$

is the probability of hitting 0. Since a martingale that is bounded must have a limit, the only other possibility is going to ∞.

$$1 - \left(\frac{q}{p}\right)^x = P[\xi_n \to \infty, \xi_n > 0 \quad \forall\, n \geq 0 | \xi_0 = x] \to 1$$

as $x \to \infty$. Therefore $c = 1$ and $P[\tau < \infty | \xi_0 = x] = (\frac{q}{p})^x$.

2.2. If $q > p$, then $\xi_n \to -\infty$ and so $P[\tau < \infty | \xi_0 = x] = 1$. Note that until it hits 0 it is just a random walk. To calculate $E[\tau]$ we note that

$$\xi_n - n(p-q)$$

is a martingale. Yields

$$E[\xi_\tau - \tau(p-q)] = x$$

But $\xi_\tau = 0$. Therefore

$$E[\tau] = \frac{x}{q-p}$$

Needs a little justification. Stop at N as well as 0. That is define

$$\tau_N = \inf[t : x(t) = 0 \text{ or } N]$$

$$x = E[\xi_{\tau_N} - \tau_N(p-q)] = 0p(x) + (1 - p_N(x))N - (p-q)E_x[\tau_N]$$

Simplifies to

$$E_x[\tau_N] = \frac{x - N(1 - p_N(x))}{(q-p)}$$

Since $p_N(x) = (\frac{p}{q})^{N-x}$, $Np_N(x) \to 0$. This completes the proof.

3.2. Let τ take values $\{s_j\}$. Let $A \in \mathcal{F}_\tau$. Need to show

$$\begin{aligned} E[f(x(t_1+\tau) - x(\tau), &x(t_2+\tau) - x(\tau), \ldots, x(t_n+\tau) - x(\tau))\mathbf{1}_A(\omega)] \\ &= P(A)E[f(x(t_1), x(t_2), \ldots, x(t_n))] \end{aligned}$$

where P is the Brownian motion probability and E is expectation with respect to P. Let $E_j = \{\omega : \tau = s_j\}$. Then $E_j \in \mathcal{F}_{s_j}$. From the independence of increments for

Brownian motion, the collection $\{x(s_j + t_i) - x(s_j)\}$ is independent of $\mathcal{F}_{t_j}$ and has the same distribution as $\{x(t_j)\}$ under P. Moreover $A \in \mathcal{F}_\tau$ means $A \cap \{\tau = t_j\} \in \mathcal{F}_{t_j}$. Hence

$$\begin{aligned} &E[f(x(t_1+\tau) - x(\tau), x(t_2+\tau) - x(\tau), \ldots, x(t_n+\tau) - x(\tau))\mathbf{1}_A(\omega)] \\ &\quad = \sum_j E[f(x(t_1+\tau) - x(\tau), x(t_2+\tau) - x(\tau), \ldots, x(t_n+\tau) - x(\tau))\mathbf{1}_{A\cap E_j}(\omega)] \\ &\quad = \sum_j E[f(x(t_1+t_j) - x(t_j), x(t_2+t_j) - x(t_j), \ldots, x(t_n+t_j) - x(t_j))\mathbf{1}_{A\cap E_j}(\omega)] \\ &\quad = \sum_j P(A \cap E_j) E[f(x(t_1), x(t_2), \ldots, x(t_n))] \\ &\quad = P(A)E[f(x(t_1), x(t_2), \ldots, x(t_n))] \end{aligned}$$

3.2. First note that $\frac{[n\tau]+1}{n} = \frac{j}{n}$ if $[n\tau] = j-1$ or $j-1 \le n\tau < j$ or $\frac{j-1}{n} \le \tau < \frac{j}{n}$. Hence the set$\omega : \frac{[n\tau(\omega)]+1}{n} = \frac{j}{n}$ is in $\mathcal{F}_{\frac{j}{n}}$ and τ_n is a stopping time. Because $\tau_n \ge \tau$, $\mathcal{F}_{\tau_n} \supset \mathcal{F}_\tau$. If $A \in \mathcal{F}_\tau$, then $A \in \mathcal{F}_{\tau_n}$ and

$$\begin{aligned} E[f(x(t_1+\tau_n) - x(\tau_n), x(t_2+\tau_n) - x(\tau_n), \ldots, x(t_k+\tau_n) - x(\tau_n))\mathbf{1}_A(\omega)] \\ = P(A)E[f(x(t_1), x(t_2), \ldots, x(t_k))] \end{aligned}$$

Assuming f to be continuous, we can let $n \to \infty$. $\tau_n \downarrow \tau$ and obtain

$$\begin{aligned} E[f(x(t_1+\tau) - x(\tau), x(t_2+\tau) - x(\tau), \ldots, x(t_k+\tau) - x(\tau))\mathbf{1}_A(\omega)] \\ = P(A)E[f(x(t_1), x(t_2), \ldots, x(t_k))] \end{aligned}$$

4.1 By Itô's formula, until time τ,

$$du(t,x) = [u_t(t,x(t)) + \frac{1}{2}u_{xx}(t,x(t))]dt + u_x(t,x(t))dx(t)$$

where $x(t)$ is Brownian Motion starting from any x with $|x| < 1$ at time s. In particular

$$u(s,x) = E_x[u(\tau \wedge t), x(\tau \wedge t))]$$

On the set $\tau \le t$, $u(\tau, x(\tau)) = u(t, \pm 1) = 0$. Hence if u is bounded by C,

$$u(s,x) \le CP[\tau > t]$$

And, by the reflection principle

$$\begin{aligned} P[\tau > t] &\le P_{s,x}[\sup_{s\le\sigma\le t} x(\sigma) \le 1] \\ &\le 1 - 2P_{s,x}[x(t) \ge 1] \\ &= 1 - 2\frac{1}{\sqrt{2\pi(t-s)}}\int_1^\infty \exp[-\frac{y^2}{2(t-s)}]dy \\ &= 2\frac{1}{\sqrt{2\pi(t-s)}}\int_0^1 \exp[-\frac{y^2}{2(t-s)}]dy \\ &\le]\frac{2}{\sqrt{2\pi(t-s)}} \\ &\to 0 \end{aligned}$$

as $t \to \infty$. As for the second part, one can construct a solution of the form

$$f(x)e^{\lambda t}$$

provided

$$\lambda f + \frac{1}{2} f_{xx} = 0.$$

$f(x) = \cos \frac{\pi}{2} x$ and $\lambda = \frac{\pi^2}{8}$ will do it.

4.2. We show that

$$u(s,x) = P[\tau < \infty \,|x(s) = 0] \to 0$$

as $s \to \infty$. By symmetry

$$P_{s,0}[\tau < \infty] \le 2P_{s,0}[\sup_{t \ge s}[x(t) - t] \ge 0]$$

If $x(t)$ is Brownian motion starting from 0 at time s, the process

$$e^{x(t) - \frac{1}{2}(t-s)}$$

is a martingale. By Doob's inequality

$$P_{s,0}[\sup_{t \ge s} e^{x(t) - \frac{1}{2}(t-s)} \ge \ell] \le e^{-\ell}$$

Take $\ell = e^{\frac{s}{2}}$. Then, and

$$P_{s,0}[\sup_{t \ge s}[x(t) - t] \ge 0] = P_{s,0}[\sup_{t \ge s} e^{x(t) - t} \ge 1] \le P_{s,0}[\sup_{t \ge s} e^{x(t) - \frac{1}{2}(t-s)} \ge \ell] \le e^{-\frac{s}{2}}$$

which is sufficient.

5.1

$$I(f) = f(T)x(T) - x(0)f(0) - \int_0^T x(s)f'(s)ds = f(T)x(T) - \int_0^T x(s)f'(s)ds$$

Clearly $I(f)$ is Gausian, has mean 0 and

$$E\big[[I(f)]^2\big] = T[f(T)]^2 + \int_0^T \int_0^T f'(t)f'(s)\min(s,t)dsdt - 2\int_0^T f(T)f'(s)\min(T,s)ds$$

This reduces to

$$\int_0^T |f(t)|^2 dt$$

if we integrate by parts. Now we approximate $f \in L_2[0,T]$ by smooth f_n and

$$\begin{aligned}\lim_{m,n\to\infty} E\big[[I(f_n) - I(f_m)]^2\big] &= \lim_{m,n\to\infty} E\big[[I(f_n) - I(f_m)]^2\big] \\ &= \lim_{m,n\to\infty} \int_0^T |f_n(t) - f_m(t)|^2 dt = 0\end{aligned}$$

$I(f_n)$ then has a limit in $L_2(P)$ and the limit $I(f)$ is clearly Gaussian with mean 0 and variance $\int_0^T |f(t)|^2 dt$.

5.2 If Z is a Gaussian random variable with mean 0 and variance σ^2, we have

$$E[|Z|] = c\sigma, \qquad E[|Z|^2] = \sigma^2, \qquad \text{Var }(|Z|) = (1-c^2)\sigma^2$$

Therefoer

$$E[V_n] = c2^n \cdot 2^{-\frac{n}{2}} = c2^{\frac{n}{2}} \qquad \text{Var }(V_n) = (1-c^2)2^n 2^{-n} = (1-c^2)$$

By Tchebechev's inequality

$$P[V_n \leq \frac{c}{2} 2^{\frac{n}{2}}] \leq P[|V_n - E(V_n)| \geq \frac{c}{2} 2^{\frac{n}{2}}] \leq \frac{4(1-c^2)}{c^2} 2^{-n}$$

Borel-Cantelli Lemma shows $V_n \to \infty$ with probability 1.

$$c = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z| e^{-\frac{z^2}{2}} dz = 2\frac{1}{\sqrt{2\pi}} \int_0^{\infty} z\, e^{-\frac{z^2}{2}} dz = 2\frac{1}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}} < 1$$