Final Examination.

Due Dec 15.

1. We have a population of size N which completely renews itself every generation. The total size is N in every generation. The population consists of two types of individuals A and B. If in a given generation the population consists of x individuals of type A and $N-x$ individuals of type B, then each member of the next generation will be of type A with probability $\frac{x}{N}$ and type B with probability $\frac{N-x}{N}$ independently of prior history. The types of different individuals are mutually independent. If X_n is the number of type A individuals in the n-th generation, then X_n is a Markov process on the finite state space consisting of $\{0, 1, 2, \ldots, N\}$.

i. What is the transition probability

$$\pi(x, y) = P[X_{n+1} = y | X_n = x] \ ?$$

and what is special when $x = 0$ or N?

ii. Show that X_n has a limit as $n \to \infty$ which is either 0 or N.

iii. If $\tau = \inf\{n : X_n = 0 \text{ or } N\}$ what is the value of

$$P[X_\tau = 0 | X_0 = x] \ ?$$

iv. Is $E[\tau] = V_N(x) < \infty$?

v. If so can you get a bound for $\sup_x V_N(x)$?

vi. Show that the distribution of $\frac{X_{[N\,t]}}{N}$ converges to a diffusion process whose generator is

$$\frac{1}{2} x(1-x) \frac{d^2}{dx^2}$$

vii. Carry out the analogs of **ii, iii, iv** in this case. Calculate $E[\tau | x(0) = x] = v(x)$.

viii. How is $V_N(x)$ related to $v(x)$?

2. $x(t)$ is the diffusion process with generator

$$\frac{1}{2} \frac{d^2}{dx^2} + b(x) \frac{d}{dx}$$

The function $b(x)$ is smooth but perhaps unbounded for large x. Moreover $b(x) < 0$ for $x > 0$ and $b(x) > 0$ for $x < 0$. Show that the process never explodes. In other words if

$$\tau_n = \inf\{t : |x(t)| \geq n\}$$

then for any $T < \infty$,

$$\lim_{n \to \infty} P[\tau_n \leq T] = 0$$