Section 12. Markov Chain Approximations

It is often necessary to approximate models in continuous time by discrete versions. The simplest example is approximation of Brownian motion by random walks. Let us consider independent random variables $X_i = \pm 1$ with probability $\frac{1}{2}$ each.

$$S_n = X_1 + X_2 + \cdots + X_n$$

and

$$\xi_n(\frac{j}{n}) = \frac{S_j}{\sqrt{n}}$$

ξ_n is interpolated linearly between $\frac{j}{n}$ and $\frac{j+1}{n}$. It is not difficult to see that the distribution of $\xi_n(t)$ converges to the normal distribution with mean 0 and variance t. In fact the joint distribution of $\{\xi_n(t_j)\}$ for any finite collection $\{t_j\}$ converges to the corresponding distribution of Brownian motion at the same times, i.e. that of $\{\beta(t_j)\}$. The proof is nothing more than the central limit theorem. However to prove

$$\lim_{n\to\infty} P[\sup_{0\le s\le t} \xi_n(s) \ge \ell] = P[\sup_{0\le s\le t} \beta(s) \ge \ell] = \frac{2}{\sqrt{2\,\pi\,t}} \int_\ell^\infty e^{-\frac{x^2}{2t}}\, dx$$

requires some work. A general sufficient condition, which allows us to make the transition from functions that depend on a finite number of coordinates to functionals that depend continuously on the trajectories (under uniform convergence) is

$$E[|\xi_n(t) - \xi_n(s)|^4] \le C|t-s|^{1+\alpha}$$

for some $\alpha > 0$ and constant C both independent of n.

More generally we can have a Markov Chain $\pi_h(x, dy)$ on R^d, that satisfy

$$\int (y_i - x_i)\pi_h(x, dy) = hb_i(x) + o(h)$$

$$\int (y_i - x_i)(y_j - x_j)\pi_h(x, dy) = ha_{i,j}(x) + o(h)$$

and

$$\int |y - x|^4 \pi_h(x, dy) = O(h^2)$$

Then the Markov Chain $\xi_h(jh) = X_h(j)$ will converge to the diffusion process with generator

$$\mathcal{A}f = \frac{1}{2}\sum a_{i,j}(x) D_{x_i} D_{x_j} + \sum_j b_j(x) D_{x_j}$$

Let us assume that the equation

$$u_t(t,x) + \frac{1}{2}\sum a_{i,j}(x)[D_{x_i} D_{x_j} u](t,x) + \sum_j b_j(x)[D_{x_j} u](t,x) = 0, u(T,x) = f(x), (T = Nh)$$

has a smooth solution. We define

$$u_h(jh, x) = E[f(X_h(Nh)|X_h(jh) = x] - u(jh, x)$$

and

$$\Delta_h(jh, x) = u_h(jh, x) - u(jh, x)$$

Then

$$\begin{aligned}
\Delta_h(jh, x) &= \int u_h((j+1)h, y)\pi_h(x, dy) - u(jh, x) \\
&= \int [u_h((j+1)h, y) - u((j+1)h, y)]\pi_h(x, dy) \\
&\quad + \int [u((j+1)h, y) - u((j+1)h, x]\pi_h(x, dy) \\
&\quad + u((j+1)h, x) - u(jh, x)
\end{aligned}$$

and

$$\begin{aligned}
\sup_x |\Delta_h(jh, x)| &\le \sup_x |\Delta_h((j+1)h, x)| \\
&\quad + \sup_x [\mathcal{A}u((j+1)h, x) + hu_t((j+1)h, x)] \\
&\quad + o(h) \\
&\le \sup_x |\Delta_h((j+1)h, x)| + o(h)
\end{aligned}$$

Proving

$$\sup_x |u_h(0, x) - u(0, x)| \le N o(h) = o(1)$$

Examples: Biased random walk. Step size $\frac{1}{\sqrt{n}}$. $X_{n+1} = X_n \pm \frac{1}{\sqrt{n}}$. But

$$\pi(j\sqrt{n}, (j \pm 1)\sqrt{n}) = \frac{1}{2} \pm \frac{b(j\sqrt{n})}{\sqrt{n}}$$

Converges to process with generator.

$$\frac{1}{2}D_{xx} + b(x)D_x$$