7. Brownian Motion as a Markov process.

As a process with independent increments given $\mathcal{F}_s$, $x(t) - x(s)$ is independent and has a normal distribution with mean 0 and variance $t - s$. Therefore

$$P[x(t) \in A|\mathcal{F}_s] = \int_A p(t-s, x(s), y)dyy$$

where

$$p(t, xy) = \frac{1}{\sqrt{2\pi(t-s)}} e^{\frac{(y-x)^2}{2(t-s)}}$$

The transition probability density $p(t, x, y)$ satisfies

$$\int p(s, x, y)p(t, y, z)dy = p(t+s, x, z)$$

and $p(t-s, x, y)$ satisfies in s, x the backward heat equation

$$p_s + \frac{1}{2}p_{xx} = 0$$

and in t, y, the forward heat equation

$$p_t = \frac{1}{2}p_{yy}$$

In particular the solution of

$$u_s + \frac{1}{2}u_{xx} = 0, u(t, x) = f(x)$$

is given by

$$u(s, x) = \int p(t-s, x, y)f(y)dy$$

and can be interpreted as

$$u(s, x) = E[f(x(t))|x(s) = x]$$

On the other hand the solution of

$$u_t = \frac{1}{2}u_{yy}, u(s, y) = g(y)$$

is solved for $t > s$ by

$$u(t, y) = \int p(t-s, x, y)g(x)dx$$

and has the interpretation as the probability density of the distribution of Brownian motion at time t when it starts form a random point $x = x(s)$ at time s, the distribution of $x(s)$ having density $g(x)$.

One can check these things by differentiating under the integral sign and the boundary condition is verifies by checking that $\int p(t,x,y)dy = 1$ and

$$\lim_{t\to 0}\int_{|x-y|\geq\epsilon} p(t,x,y)dy = 0$$

There is an important connection between Brownian motion and the operator $\delta_t + \frac{1}{2}\delta_x^2$ besides the ones described above.

Fact. Let $u(t,x)$ be a nice function. Smooth and well behaved at ∞. Let $f(t,x) = u_t + \frac{1}{2}u_{xx}$. Then $u(t,x(t)) - \int_s^t f(\sigma,x(\sigma))d\sigma$ is a maringale with respect to Brownian motion starting from any point x at time s.

Proof. We need to prove

$$E[u(t,x(t)) - u(s,(x(s)) - \int_s^t f(\sigma,x(\sigma))d\sigma|\mathcal{F}_\sigma] = 0$$

This is just verifying

$$\int u(t,y)p(t-s,x,y)dy - u(s,x) - \int_s^t\int f(\sigma,y)p(\sigma-s,x,y)dyd\sigma = 0$$

It is enough to check for functions of the form $u(t,x) = g(t)e^{i\xi x}$. Then

$$f(t,x) = [g'(t) - \frac{\xi^2}{2}g(t)]e^{i\,\xi\,x}$$

Note that

$$\int e^{i\,\xi\,y}p(t,x,y) = e^{i\,\xi\,x}e^{-\frac{\xi^2 t}{2}}$$

We need to check

$$e^{i\,\xi\,x}e^{-\frac{\xi^2(t-s)}{2}}g(t) - e^{i\,\xi\,x}g(s) = \int_s^t e^{i\,\xi\,x}[g'(\sigma) - g(\sigma)\frac{\xi^2}{2}]e^{-\frac{\xi^2(\sigma-s)}{2}}d\sigma$$

or

$$e^{-\frac{\xi^2(t-s)}{2}}g(t) - g(s) = \int_s^t [g'(\sigma) - g(\sigma)\frac{\xi^2}{2}]e^{-\frac{\xi^2(\sigma-s)}{2}}d\sigma$$

which is easily carried out. A consequence is the maximum principle. For any solution u of $u_t + \frac{1}{2}u_{xx} = 0$, $u(t,x(t))$ is a martingale. In particular

$$u(s,x) = E[u(t,x(t))|\mathcal{F}_s] = \int u(t,y)p(t-s,x,y)dy$$

If $u(t,y) \geq 0$, then so is $u(s,y)$ for $s \leq t$. In particular this proves the uniqueness of solutions.

One can have functions defined in a region. For instance $s \leq t \leq T, |x| \leq 1$. Then $\tau = \min\{T, \inf\{t : |x(t)| \geq 1\}\}$ is a stopping time and

$$E[u(\tau, x(\tau)) - u(s, x(s)) - \int_s^\tau f(\sigma, x(\sigma))d\sigma | x(s) = x] = 0$$

There are multi dimensional versions of Brownian motion. Just take independent versions $\{x_j(t); 1 \leq j \leq d\}$ to get the d-diemsional version.

$$p(t, \{x_j\}, \{y_j\}) = \Pi_{j=1}^d p(t, x_j, y_j)$$

The backward differential equation is

$$\frac{\partial}{\partial t} + \frac{1}{2}\Delta = \frac{\partial}{\partial t} + \frac{1}{2}\sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$$