8. Stochastic Differential Equations.

Brownian motion has the property that the distribution of $x(t+h)-x(t)$ given the σ-field $\mathcal{F}_t$ of information up to time t, is Gaussian with mean 0 and variance h. One can visualize a Markov process $x(t)$ for which the corresponding conditional distribution of the increment is approximately Gaussian with mean $hb(t,x(t))$ and variance $ha(t,x(t))$. The Markovian nature is reflected in that the conditional distribution depends only on $t, x(t)$ and not on the information about $x(s)$ for all times $s \le t$. $x(t)$ can be one dimensional or d-dimensional. In such a case $b(t,x)$ would be a d-dimensional vector and $a(t,x)$ would be a symmetric positive semi-definite matrix $ha(t,x(t))$ would then be approximately the conditional variance covariance matrix of the vector $x(t+h)-x(t)$. If we do have a $p(s,x,t,dy)$ which are the transition probabilities of our Markov process,

$$b(t,x) = \lim_{h\to 0}\frac{1}{h}\int (y_i - x_i)p(t,x,t+h,dy)$$

and

$$a_{i,j}(t,x) = \lim_{h\to 0}\frac{1}{h}\int (y_i - x_i)(y_j - x_j)p(t,x,t+h,dy) \tag{1}$$

One has to be a little careful. The moments may not exist. May be safer to truncate.

$$b(t,x) = \lim_{h\to 0}\frac{1}{h}\int_{\{y:\|y-x|\le 1\}} (y_i - x_i)p(t,x,t+h,dy) \tag{2}$$

and

$$a_{i,j}(t,x) = \lim_{h\to 0}\frac{1}{h}\int_{\{y:\|y-x\|\le 1\}} (y_i - x_i)(y_j - x_j)p(t,x,t+h,dy)$$

Why 1? Should not matter. For any $\epsilon > 0$,

$$\lim_{h\to 0}\frac{1}{h}\int_{\{y:\|y-x\|\ge \epsilon\}} p(t,x,t+h,dy) = 0 \tag{3}$$

Or one can assume that for some $\delta > 0$,

$$\lim_{h\to 0}\frac{1}{h}\int \|y-x\|^{2+\delta}p(t,x,t+h,dy) = 0$$

which will imply (3) and avoid the necessity of truncation in (1) and (2).

One way to generate a Gaussian random vector Y with mean hb and covariance ha is to start from a Gaussian vector X with mean 0 and covariance hI and do a linear transformation.

$$Y = hb + \sigma X$$

where $\sigma\sigma^* = a$. Then $E[Y] \simeq hb$ and $Var\langle \theta, Y\rangle = Var\langle \theta, \sigma Y\rangle = Var\langle \sigma^*\theta, Y\rangle = h\|\sigma^*\theta\|^2 = h\langle \theta\sigma\sigma^*\theta\rangle = h\langle \theta, a\theta\rangle$. One can choose σ to be the positive semi-definite symmetric square root of a.

This leads to the possible definition

$$dx(t) = b(t, x(t))dt + \sigma(t, x(t))d\beta(t); x(0) = x$$

Or we can write it in integrated form

$$x(t) = x + \int_0^t \sigma(s, x(s))d\beta(s) + \int_0^t b(s, x(s))ds \tag{4}$$

We are given functions $b(t, x) : [0, T] \times R^d \to R^d$ and $\sigma(t, x) : [0, T] \times R^d \to M_{d\times d}$. d dimensional Brownian motion $\beta(t)$ on $(\Omega, \mathcal{F}_t, P)$. We want to existence and uniqueness of progressively measurable, almost surely continuos functions $x(t, \omega)$ that satisfy (4). The assumptions that we will make on b and σ are the following.

$$\|b(t, x) - b(t, y)\| \le C\|x - y\|; \|\sigma(t, x) - \sigma(t, y)\| \le C\|x - y\|$$

and

$$\|b(t, x)\| \le C, \quad \|\sigma(t, x)\| \le C$$

First let us prove uniqueness. Let there be two solutions $x(t)$ and $y(t)$. Then

$$x(t) - y(t) = \int_0^t [\sigma(s, x(s)) - \sigma(s, y(s))]d\beta(s) + \int_0^t [b(s, x(s)) - b(s, y(s))]ds$$

Since σ and b are bounded $x(t)$ and $y(t)$ have finite second moments. Denoting by $\Delta(t) = E[\|x(t) - y(t)\|^2]$,

$$\begin{aligned}
\Delta(t) &\le 2E\Big[[\int_0^t [\sigma(s, x(s)) - \sigma(s, y(s))]d\beta(s)]^2\Big] + 2E\Big[[\int_0^t [b(s, x(s)) - b(s, y(s))]ds]^2\Big] \\
&\le 2E\Big[\int_0^t [\|\sigma(s, x(s)) - \sigma(s, y(s))\|^2 ds\Big] + 2TE\Big[\int_0^t \|b(s, x(s)) - b(s, y(s))\|^2 ds\Big] \\
&\le C(T)\int_0^t \Delta(s)ds
\end{aligned}$$

It is easy to see that $\Delta(t) \le C(T)$. By induction

$$\Delta(t) \le \frac{[C(T)]^{n+1}t^n}{n!}$$

for every n and is therefore 0.

Now we turn to existence. We take $x_0(t) \equiv x$ and then by induction define

$$x_n(t) = x + \int_0^t \sigma(s, x_{n-1}(s))d\beta(s) + \int_0^t b(s, x_{n-1}(s))ds \tag{5}$$

Denoting by $z_n(t)$ the difference $x_n(t) - x_{n-1}(t)$, we have

$$z_{n+1}(t) = \int_0^t [\sigma(s, x_n(s)) - \sigma(s, x_{n-1}(s))]d\beta(s) + \int_0^t [b(s, x_n(s)) - b(s, x_{n-1}(s))]ds$$

Denoting by $\Delta_n(t) = E[\|z_n(t)\|^2]$ we have just as in the proof of uniqueness

$$\Delta_{n+1}(t) \le C(T) \int_0^t \Delta_n(s)ds$$

with $\Delta_1(t) \le C(T)$. Again by induction,

$$\Delta_{n+1}(t) \le [C(T)]^{n+1} \frac{t^n}{n!}$$

This implies

$$\sum_{n=1}^{\infty} \sup_{0 \le t \le T} \sqrt{\Delta_n(t)} < \infty$$

By Doob's inequality for martingales for the stochastic integral term and stand estimate for the ordinary intgral

$$E[\sup_{0 \le t \le T} \|z_{n+1}(t)\|^2] \le C(T) \int_0^T \Delta_n(t)dt$$

proving that $\sum z_n(t)$ converges uniformly with probability 1. Therefore

$$x(t) = \lim_{n \to \infty} x_n(t)$$

exists and satisfies (4). The same estimates allow us to compare two solutions $x(t)$, $y(t)$ for the same equation but starting from different points x and y.

$$E[\sup_{0 \le t \le T} \|x(t) - y(t)\|^2] \le C(T)\|x - y\|^2$$

We can start from $x(s) = x$ and solve for $t \ge s$. We denote by $p(s, x, t, A)$ the probability

$$p(s, x, t, A) = P[x(t) \in A | x(s) = x]$$

We can start from $x(s) = x(s, \omega)$ a random variable that is $\mathcal{F}_s$ measurable and nothing would change. If we stop at s and then find a new solution for $t \ge s$ starting from $x(s, \omega)$

by uniqueness this is the same as $x(t,\omega)$. The conditional distribution of $x(t,\omega)$ given $\mathcal{F}_s$ is the same as the distribution of a solution starting from $x(s,\omega)$ at time s.

$$P[x(t) \in A|\mathcal{F}_s] = p(s, x(s,\omega), t, A)$$

which proves the Markov property. The strong Markov property is similar and we use the strong Markov property of the Brownian motion. It is not hard now to prove that (1) and (2) hold.

$$\frac{1}{h}E\|x(h) - b(0,x)h] = o(1) ash \to 0$$

as is

$$\frac{1}{h}E\|x(h) - b(0,x)h - \sigma(0,h)\beta(h)\|^2]$$

Exponential Martingales. For any stochastic integral

$$x(t) = \int e(s,\omega)d\beta(s)$$

with bounded σ

$$\exp[x(t) - \frac{1}{2}\int_0^t \|e(s,\omega)\|^2 ds]$$

is a martingale. Star with simple functions. Pass to the limit. Get bounds to prove uniform integrability. If $x_n(t)$ is an approximation using a simple function $e_n(s,\omega)$ having the same bound C as $e(s,\omega)$,

$$E\Big[\exp\big[2[x_n(t) - \frac{1}{2}\int_0^t \|e_n(s,\omega)\|^2 ds]\big]\Big] \le E\Big[\exp\big[[2x_n(t) - \frac{1}{2}\int_0^t \|2e_n(s,\omega)\|^2 ds] + Ct\big]\Big]$$
$$\le e^{Ct}$$

and this is enough. For any solution $x(t)$ of SDE (4) and any $\theta \in R^d$

$$\exp[\langle\theta, x(t) - x(0)\rangle - \int_0^t \langle\theta, b(s,x(s))\rangle ds - \frac{1}{2}\int_0^t \langle\theta, a(s,x(s))\theta\rangle ds]$$

is a martingale.