

1. Find all eigenvectors of the matrix

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

2. Let (e_1, e_2, e_3) be basis of V and $L : V \rightarrow V$ be the map given by

$$\begin{aligned} L(e_1) &= e_1 + e_2 \\ L(e_2) &= 2e_2 + e_3 \\ L(e_3) &= 2e_3. \end{aligned}$$

Find the eigenvectors of L by solving

$$L(v) = \lambda v,$$

where

$$v = e_1 a^1 + e_2 a^2 + e_3 a^3,$$

directly for a^1, a^2, a^3, λ .

3. True or false? Explain. If false, provide a counterexample.

3.1. If a matrix has one eigenvector, then it has infinitely many eigenvectors.

3.2. Every matrix over a field $\mathbb{F}$ has at least one eigenvalue in $\mathbb{F}$.

3.3. Similar matrices always have the same eigenvectors.

3.4. If two matrices have the same eigenvalues, then they are similar

4. Suppose, for a matrix A , v_1 is an eigenvector for the eigenvalue λ_1 and v_2 is an eigenvector for the eigenvalue λ_2 . What are the necessary and sufficient conditions that $v_1 + v_2$ be an eigenvector of A and what is its eigenvalue?

5. For any $\alpha \in \mathbb{R}$, the real matrix

$$R_\alpha = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

is called a rotation matrix, because it rotates each vector counterclockwise by the angle α . Compute the (possibly complex) eigenvalues and eigenvectors of R_α

6. A linear transformation $A : V \rightarrow V$ is called **nilpotent** if for some positive integer k , A^k is the zero matrix. Prove that the only eigenvalue of A is 0.