

MA-GY7033 Linear Algebra I

Eigenvalues and Eigenvectors

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Outline

1. Eigenvalues and the Characteristic Polynomial
2. Complex Vector Spaces
3. Diagonalization

Eigenvalues and Eigenvectors of a Linear Transformation

- ▶ Consider a linear map $L : V \rightarrow V$
- ▶ A scalar $\lambda \in \mathbb{R}$ is called an **eigenvalue** of L if there is a nonzero vector $v \in V$ such that any of the following equivalent statements hold

$$L(v) = \lambda v \iff (L - \lambda I)v = 0 \iff v \in \ker(L - \lambda I)$$

The vector v is called an **eigenvector**

- ▶ λ is an eigenvalue of $L \iff$ the linear map $L - \lambda I$ is singular
- ▶ Any nonzero $v \in \ker(L - \lambda I)$ is an eigenvector for the eigenvalue λ
- ▶ Any nonzero $v \in \ker L$ is an eigenvector for the eigenvalue 0
- ▶ The eigenspace for an eigenvalue λ of L is the subspace

$$E_\lambda(L) = \ker(L - \lambda I) = \{v \in V : L(v) = \lambda v\}$$

A nonzero vector v is an eigenvector for the eigenvalue λ if and only if $v \in E_\lambda(L)$

Eigenvalues and Eigenvectors of a Square Matrix

- ▶ A scalar λ is an **eigenvalue** of a matrix $M \in \mathcal{M}_{n \times n}(\mathbb{F})$ if there is a nonzero vector $a \in \mathbb{F}^n$ such that

$$Ma = \lambda a$$

The vector $a \in \mathbb{F}^n$ is called an **eigenvector**

- ▶ λ is an eigenvalue of M if and only if the matrix $M - \lambda I$ is singular
- ▶ Therefore,

$$\lambda \text{ is an eigenvalue of } M \iff \det(M - \lambda I) = 0$$

Eigenvalues and Eigenvectors With Respect to Basis

- ▶ If $(e_1, \dots, e_n)$ is a basis of V , then there is a matrix M such that $L(e_k) = e_1 M_k^1 + \dots + e_n M_k^n$
- ▶ If $v = e_1 a^1 + \dots + e_n a^n$ is an eigenvector with eigenvalue λ , then on one hand

$$\begin{aligned} L(v) &= L(e_1 a^1 + \dots + e_n a^n) \\ &= L(e_1) a^1 + \dots + L(e_n) a^n \\ &= (e_1 M_1^1 + \dots + e_n M_1^n) a^1 + \dots + (e_1 M_n^1 + \dots + e_n M_n^n) a^n \\ &= e_1 (M_1^1 a^1 + \dots + M_n^1 a^n) + \dots + e_n (M_1^n a^1 + \dots + M_n^n a^n) \\ &= \begin{bmatrix} e_1 & \dots & e_n \end{bmatrix} \begin{bmatrix} M_1^1 & \dots & M_n^1 \\ \vdots & \vdots & \vdots \\ M_1^n & \dots & M_n^n \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix} \\ &= E(Ma) \end{aligned}$$

but $L(v) = \lambda v = \lambda E a = E(\lambda a)$

- ▶ Therefore, a is an eigenvalue of the matrix M

Change of Basis Formulas

- ▶ Let $E = [e_1 \ \cdots \ e_n]$ and $F = [f_1 \ \cdots \ f_n]$ be bases such that $F = EH$

- ▶ If

$$L(E) = EM \text{ and } L(F) = FN,$$

then

$$FN = L(F) = L(EH) = L(E)H = EMH = FH^{-1}MH$$

and therefore $N = H^{-1}MH$

- ▶ If $v = Ea = Fb = Ehb$ and therefore $a = Hb$.
- ▶ Therefore, $b = H^{-1}a$ is an eigenvector of $N = H^{-1}MH$ if and only if a is an eigenvector of M

Determinants, Eigenvalues, and Eigenvectors of Similar Matrices

- ▶ Two matrices M and N are called **similar** if there is an invertible matrix H such that

$$M = HNH^{-1}, \text{ i.e., } N = H^{-1}MH$$

- ▶ In particular, $\det(M) = \det(N)$
- ▶ And a is an eigenvector of M with eigenvalue λ if and only if $b = H^{-1}a$

Characteristic Polynomial of a Matrix

- ▶ Let δ_k^j be the element in the j -th row and k -column of the identity matrix, i.e.,

$$\delta_k^j = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

- ▶ Observe that the function $p_M : \mathbb{F} \rightarrow \mathbb{F}$ given by

$$\begin{aligned} p_M(\lambda) &= \det(M - \lambda I) \\ &= \sum_{\sigma \in S_n} \epsilon(\sigma) (M - \lambda I)_1^{\sigma(1)} \cdots (M - \lambda I)_n^{\sigma(n)} \\ &= \sum_{\sigma \in S_n} \epsilon(\sigma) (M_1^{\sigma(1)} - \lambda \delta_1^{\sigma(1)}) \cdots (M_n^{\sigma(n)} - \lambda \delta_n^{\sigma(n)}) \end{aligned}$$

is a polynomial in λ of degree n

- ▶ The roots of p_M are eigenvalues of M

Characteristic Polynomial of a Linear Transformation

- ▶ Recall that if $(e_1, \dots, e_n)$ is a basis of V and $A \in \Lambda^n V^*$ satisfies $A(e_1, \dots, e_n) = 1$, then

$$\begin{aligned} \det(L - \lambda I) &= A((L - \lambda I)e_1, \dots, (L - \lambda I)e_n) \\ &= A(L(e_1) - \lambda e_1, \dots, L(e_n) - \lambda e_n) \\ &= A(L(e_1), \dots, L(e_n)) \\ &\quad - \lambda(A(e_1, L(e_2), \dots, L(e_n)) + \dots + A(L(e_1), \dots, L(e_{n-1}), e_n)) \\ &\quad + \dots + (-\lambda)^{n-1}(A(L(e_1), e_2, \dots, e_n) + \dots + A(e_1, \dots, e_{n-1}, L(e_n))) \\ &\quad + (-\lambda)^n A(e_1, \dots, e_n), \end{aligned}$$

which is a polynomial of degree n

- ▶ On the other hand, if $L(E) = EM$, then $(L - \lambda I)(E) = E(M - \lambda I)$ and therefore

$$p_L(\lambda) = \det(L - \lambda I) = \det(M - \lambda I) = p_M(\lambda)$$

Similar Matrices Have the Same Characteristic Polynomial

- ▶ If $M = HNH^{-1}$, then

$$M - \lambda I = HNH^{-1} - \lambda H(I)H^{-1} = H(N - \lambda I)H^{-1}$$

- ▶ Therefore,

$$\det(M - \lambda I) = (\det H)(\det(N - \lambda I))(\det H^{-1}) = \det(N - \lambda I)$$

Eigenvalues, Eigenvectors, Characteristic Polynomial of a Diagonal Matrix

- ▶ A matrix is **diagonal** if its only nonzero entries are on the diagonal, e.g.,

$$D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

- ▶ It is easy to see that the eigenvalues are a, b, c
- ▶ It is also easy to see that each standard basis vector is an eigenvector
- ▶ The characteristic polynomial is

$$\begin{aligned} p_D(x) &= \det(D - xI) \\ &= \det \begin{bmatrix} a - x & 0 & 0 \\ 0 & b - x & 0 \\ 0 & 0 & c - x \end{bmatrix} \\ &= (a - x)(b - x)(c - x) \end{aligned}$$

Characteristic Polynomial of a Triangular Matrix

- ▶ A matrix is **lower triangular** if entries above the diagonal are zero, e.g.,

$$T = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$$

- ▶ The characteristic polynomial is

$$\begin{aligned} p_T(x) &= \det(T - xI) \\ &= \det \begin{bmatrix} a-x & 0 & 0 \\ b & c-x & 0 \\ d & e & f-x \end{bmatrix} \\ &= (a-x)(c-x)(f-x) \end{aligned}$$

- ▶ The roots of p_T are a, c, f

Eigenvalues and Eigenvectors of a Triangular Matrix

- ▶ The eigenvalues of

$$T = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$$

are therefore a, c, f

- ▶ It is easy to find an eigenvector for the eigenvalue f ,

$$Te_3 = fe_3$$

- ▶ It takes more work to find the eigenvectors for a and c

Eigenvalues and Eigenvectors Via Row Reduction (Part 1)

- ▶ Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- ▶ We can find $\ker A - \lambda I$ using row reduction

$$\begin{aligned} \ker(A - \lambda I) &= \ker \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \\ &= \ker \begin{bmatrix} 1 & -\lambda \\ -\lambda & 1 \end{bmatrix} \\ &= \ker \begin{bmatrix} 1 & -\lambda \\ 0 & 1 - \lambda^2 \end{bmatrix} \end{aligned}$$

- ▶ If $1 - \lambda^2 \neq 0$, then we can row reduce one more time to get

$$\ker(A - \lambda I) = \ker \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \{(0, 0)\}$$

Eigenvalues and Eigenvectors Via Row Reduction (Part 2)

- ▶ If $1 - \lambda^2 = 0$, then there are two cases
- ▶ $\lambda = 1$:

$$\ker(A - \lambda I) = \ker \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \{(t, t) : t \in \mathbb{F}\}$$

- ▶ $\lambda = -1$:

$$\ker(A - \lambda I) = \ker \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \{(t, -t) : t \in \mathbb{F}\}$$

- ▶ Conclusion:
 - ▶ The eigenvalues of A are $-1, 1$
 - ▶ The eigenspace for the eigenvalue -1 is $\{(t, -t) : t \in \mathbb{F}\}$
 - ▶ The eigenspace for the eigenvalue 1 is $\{(t, t) : t \in \mathbb{F}\}$

Eigenvalues and Eigenvectors Via Row Reduction (Part 3)

- Consider the matrix

$$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- We can find $\ker B - \lambda I$ using row reduction

$$\begin{aligned} \ker(B - \lambda I) &= \ker \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} \\ &= \ker \begin{bmatrix} 1 & -\lambda \\ -\lambda & -1 \end{bmatrix} = \ker \begin{bmatrix} 1 & -\lambda \\ 0 & -1 - \lambda^2 \end{bmatrix} \end{aligned}$$

- Since $-1 - \lambda^2 < 0$ for every real λ , we can always row reduce to get

$$\ker(B - \lambda I) = \ker \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \{(0, 0)\}$$

which means that B has no real eigenvalues

- Or, if $\mathbb{F} = \mathbb{C}$, then the eigenvalues of $i, -i$

Complex Numbers

- ▶ Recall that a complex number is written as $z = x + iy$, where $x, y \in \mathbb{R}$ and i is a symbol or variable that satisfies

$$i^2 = -1$$

- ▶ The set of all complex numbers is denoted $\mathbb{C}$
- ▶ The rules of complex arithmetic are the same as for polynomials in the variable i except that powers of i can be reduced to ± 1 or $\pm i$
- ▶ If $z = x + iy$
 - ▶ x is called the real part of z and y the imaginary part
 - ▶ $\bar{z} = x - iy$ is called the **conjugate** of z
 - ▶ We define $|z| \in [0, \infty)$ by $|z|^2 = z\bar{z} = x^2 + y^2$
 - ▶ The reciprocal of z is given by

$$\frac{1}{z} = \frac{\bar{z}}{\bar{z}z} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}$$

Back to Example

- ▶ Consider the matrix

$$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- ▶ We showed that

$$\ker(B - \lambda I) = \ker \begin{bmatrix} 1 & -\lambda \\ 0 & -1 - \lambda^2 \end{bmatrix}$$

- ▶ If $\lambda \in \mathbb{C}$ is an eigenvalue of B , then

$$0 = 1 + \lambda^2 = (\lambda - i)(\lambda + i)$$

and therefore the eigenvalues are $i, -i$

- ▶ $v = (v^1, v^2) \in \mathbb{C}^2$ is an eigenvector for the eigenvalue i if

$$v^1 - i v^2 = 0 \text{ or, equivalently, } v = (ic, c) \text{ where } c \in \mathbb{C}$$

- ▶ $v = (v^1, v^2) \in \mathbb{C}^2$ is an eigenvector for the eigenvalue $-i$ if

$$v^1 + i v^2 = 0 \text{ or, equivalently, } v = (-ic, c) \text{ where } c \in \mathbb{C}$$

Eigenvalues of a Complex Matrix

- ▶ Reminder: A real matrix is a complex matrix
- ▶ The eigenvalues of an n -by- n matrix M are the roots of the characteristic polynomial p_M
- ▶ The characteristic polynomial p_M has degree n
- ▶ Fundamental Theorem of Algebra: Any polynomial

$$p(z) = a_n z^n + \cdots + a_0, \quad a_n, \cdots, a_0 \in \mathbb{C}, \text{ where } a_n \neq 0,$$

can be factored into linear factors

$$p(z) = a_n(z - \lambda_1) \cdots (z - \lambda_n),$$

where $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ are the roots of p

- ▶ It follows that there are always n complex (possibly real) eigenvalues, but an eigenvalue might appear more than once
- ▶ The multiplicity of an eigenvalue λ is the number of factors equal to $(z - \lambda)$

Basic Example 1

- ▶ The characteristic polynomial of

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is

$$p_I(\lambda) = (1 - \lambda)^2$$

- ▶ The only eigenvalue of I is 1, which has multiplicity 2
- ▶ Every nonzero vector in $\mathbb{F}^2$ is an eigenvector
- ▶ The space of eigenvectors spans a 2-dimensional subspace

Basic Example 2

- ▶ The characteristic polynomial of

$$J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

is also $p_J(\lambda) = (1 - \lambda)^2$

- ▶ The only eigenvalue of J is 1 which has multiplicity 2
- ▶ A nonzero vector $v = (v^1, v^2)$ is an eigenvector if and only if

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (J - I)v = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} v^2 \\ 0 \end{bmatrix}$$
$$\implies v = \begin{bmatrix} 0 \\ c \end{bmatrix}, \text{ where } c \in \mathbb{C}$$

- ▶ The space of eigenvectors spans a 1-dimensional subspace
- ▶ In general, the space of eigenvectors for an eigenvalue λ spans a subspace whose dimension is less than or equal to the multiplicity of λ

Diagonalizable Matrix (Part 1)

- ▶ A square matrix M is **diagonalizable** if it is similar to a diagonal matrix
- ▶ Not all matrices are diagonalizable
- ▶ Suppose $M = SDS^{-1}$, where D is a diagonal matrix,

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

- ▶ The eigenvalues $\lambda_1, \dots, \lambda_n$ of D are the elements along the diagonal
- ▶ Each vector in the standard basis of $\mathbb{F}^n$ is an eigenvector of D
- ▶ Since S is invertible, its columns

$$(c_1, \dots, c_n) = (Se_1, \dots, Se_n)$$

also form a basis of $\mathbb{F}^n$

Diagonalizable Matrix (Part 2)

- ▶ If $M = SDS^{-1}$, where D is diagonal, then the columns of S

$$(c_1, \dots, c_n) = (Se_1, \dots, Se_n)$$

are a basis of $\mathbb{F}^n$ such that

$$Mc_k = MSe_k = SDS^{-1}Se_k = SDe_k = S(\lambda_k e_k) = \lambda_k Se_k = \lambda_k c_k$$

- ▶ Therefore, the columns of S form a basis comprised of eigenvectors
- ▶ Conversely, if $(b_1, \dots, b_n)$ is a basis of $\mathbb{F}^n$ comprised of eigenvectors of M , then $M = SDS^{-1}$, where the columns of S are $(b_1, \dots, b_n)$ and D is the diagonal matrix with the respective eigenvalues
- ▶ Theorem: *An n -by- n matrix M is diagonalizable if and only if there exists a basis of $\mathbb{F}^n$ comprised of eigenvectors of M*

Diagonalizable Linear Map

- ▶ A linear map $L : V \rightarrow V$, where V is a finite dimensional vector space, is **diagonalizable** if the following holds:

- ▶ There exists a basis $(e_1, \dots, e_n)$ such that each e_k is an eigenvector of L ,

$$L(e_k) = \lambda_k e_k$$

- ▶ The matrix associated with this basis is the diagonal matrix with $\lambda_1, \dots, \lambda_n$ along the diagonal
- ▶ Equivalently, L is diagonalizable if the matrix M with respect to **any** basis of V is similar to a diagonal matrix

Not Every Linear Transformation is Diagonalizable

- ▶ Consider the linear map $L : V \rightarrow V$, where if (e_1, e_2) is a basis,

$$L(e_1) = e_2$$

$$L(e_2) = 0$$

- ▶ If $v = ae_1 + be_2$, then

$$L(v) = L(ae_1 + be_2) = ae_2$$

- ▶ Therefore v is an eigenvector if and only if $a = 0$
- ▶ It follows that the only eigenvalue is 0 and the only eigenvectors are ce_2 , where $c \in \mathbb{F}$
- ▶ It follows that L is not diagonalizable

Not Every Matrix is Diagonalizable

- ▶ If

$$M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } v = \begin{bmatrix} a \\ b \end{bmatrix},$$

then

$$Mv = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

- ▶ It follows that v is an eigenvector only if $b = 0$
- ▶ The only eigenvalue is 0