MATH-GA2450 Complex Analysis

Course Information
Mathematical Grammar and Logic
Complex Numbers
Polar Coordinates
Complex Functions

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Course Information

- Web Pages
  - My homepage: https://math.nyu.edu/~yangd
  - Course Homepage
  - Course Calendar
- Textbook
  - Serge Lang, **Complex Analysis**
  - You can download the PDF for free, if your computer is connected to the NYU network. You can also buy a softcover edition for $39.99 from Springer

Prerequisites: Mathematical Grammar

- Always write in complete English or mathematical sentences
- A sentence must have a subject and verb
- A mathematical sentence usually contains an object
- Sample sentences
  - *(subject)* **equals** *(object)*
  - *(subject)* **is less than** *(object)*
  - **If** *(sentence)*, **then** *(sentence)*
  - **There exists** *(object)***such that** *(sentence)*
  - **For any** *object*, **(sentence)**

Prerequisites: Basic Deductive Logic

- You are expected to know how to use deductive logic
- Suppose A and B are English or mathematical sentences
- You are expected to know the meaning of the following phrases:

$$
\begin{aligned}
&A \text{ and } B \\
&A \text{ or } B \\
&A \text{ is false, i.e., not } A \\
\text{If } &A\text{, then } B\text{, i.e., } A \implies B \\
&A \text{ if and only if } B\text{, i.e., } A \iff B
\end{aligned}
$$

Converse and Contrapositive

- The **converse** of the sentence

  $$A \implies B$$

  is

  $$B \implies A$$

  These two are **not** equivalent
- The **contrapositive** of the sentence

  $$A \implies B$$

  is

  $$(\text{not } B) \implies (\text{not } A)$$

  These two **are** equivalent

Quantifiers

- You are expected to know the meaning of

  For each *(object)*, *(sentence)*

  which can also be written as

  $$\forall \textit{(object)}, \textit{(sentence)}$$

- You are expected to know the meaning of

  There exists *(object)*, such that *(sentence)*

  which can also be written as

  $$\exists \textit{(object)}, \textit{(sentence)}$$

Negations

- The negation of A and B is

$$(\text{not } A) \text{ or } (\text{ not } B)$$

- The negation of A or B is

$$(\text{not } A) \text{ and } (\text{ not } B)$$

- The negation of if A, then B is

$$A \text{ and } (\text{ not } B)$$

- The negation of $\forall$*(object)*, *(sentence)* is

$$\exists \textit{(object)}, \text{ such that } \textit{(negation of sentence)}$$

- The negation of $\exists$*(object)* such that *(sentence)* is

$$\forall \textit{(object)}, \textit{(negation of sentence)}$$

Modus Ponens

- All calculations and proofs **must** proceed as follows:
  - Known to be true (by definition, assumption, or proof)
    - A
    - $A \implies B$
  - True by deduction
    - B

Definitions Versus Theorems

- **VERY VERY IMPORTANT:** When studying or doing problems, make sure you know the definitions of every word and symbol
- Always try to solve problem (e.g., doing a proof) using **ONLY** definitions
- Use a theorem **ONLY** if absolutely necessary

Complex Numbers

- Set of complex numbers is denoted $\mathbb{C}$
- A complex number is denoted

$$z = x + iy,$$

  where $x, y \in \mathbb{R}$
- Complex addition and multiplication
  - Treat i as a variable
  - Do calculation using standard rules of polynomial algebra
  - Simplify using the assumption that $i^2 = -1$

Properties of Complex Addition and Multiplication

- Addition is associative and commutative
- Multiplication is associate and commutative
- $0 = 0 + i0$ is the identity element for addition
- The additive inverse of $z = x + iy$ is

$$-z = -x - iy$$

- $1 = 1 + i0$ is the identity element for multiplication

Examples

$$
\begin{aligned}
(2+i3)+(4-i) &= 2+4+i(3-1) \\
&= 6+i2 \\
(2+i3)(-1+i) &= 2(-1)+2i+i3(-1)+i3(i) \\
&= -2+2i-3i+i^2 3 \\
&= -2-3+2i-3i \\
&= -1-i \\
(x+iy)+(u+iv) &= (x+u) \\
(x+iy)(u+iv) &= xu+x(iv)+iy(u)+(iy)(iv) \\
&= xu+ixv+iyu+i^2 yv \\
&= (xu-yv)+i(xv+yu)
\end{aligned}
$$

Conjugate of Complex Number

- Conjugate of a complex number $z = x + iy$ is

$$\bar{z} = x - iy$$

- For each $z, w \in \mathbb{C}$,

$$\overline{z + w} = \bar{z} + \bar{w}$$
$$\overline{zw} = \bar{z}\bar{w}$$
$$\bar{\bar{z}} = z$$

Magnitude of Complex Number

- Magnitude squared of z is

$$|z|^2 = z\bar{z} = (x + iy)(x - iy) = x^2 - (iy)^2 = x^2 + y^2$$

- Since $|z|^2$ is always real, can define magnitude of z to be

$$|z| = \sqrt{|z|^2}$$

- Examples
  - $|2 + i3|^2 = 2^2 + 3^2 = 13$
  - If $z, w \in \mathbb{C}$,

$$\begin{aligned}
|zw|^2 &= zw\overline{zw} \\
&= zw\bar{z}\bar{w} \\
&= z\bar{z}w\bar{w} \\
&= |z|^2|w|^2 \\
|z + w|^2 &= (z + w)(\bar{z} + \bar{w}) \\
&= z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w} \\
&= |z|^2 + z\bar{w} + \bar{z}w + |w|^2
\end{aligned}$$

Example of Complex Division

$$
\begin{aligned}
\frac{2+i3}{1-i4} &= \left(\frac{2+i3}{1-i4}\right)\left(\frac{1+i4}{1+i4}\right) \\
&= \frac{(2+i3)(1+i4)}{(1-i4)(1+i4)} \\
&= \frac{2-12+i(3+8)}{1+16} \\
&= \frac{-10+i11}{17} \\
&= \frac{-10}{17} + i\frac{11}{17}
\end{aligned}
$$

Complex Division

- If $z = x + iy$ and $w = u + iv \neq 0$, then

$$\begin{aligned} \frac{z}{w} &= \frac{x+iy}{u+iv} \\ &= \left(\frac{x+iy}{u+iv}\right)\left(\frac{u-iv}{u-iv}\right) \\ &= \frac{(x+iy)(u-iv)}{(u+iv)(u-iv)} \\ &= \frac{xu+yv+i(yu-xv)}{u^2+v^2} \\ &= \frac{xu+yv}{u^2+v^2} + i\frac{yu-xv}{u^2+v^2} \end{aligned}$$

- Equivalently,

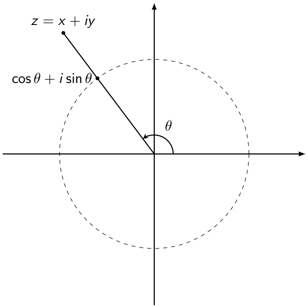
$$\begin{aligned} \frac{z}{w} &= \left(\frac{z}{w}\right)\left(\frac{\bar{w}}{\bar{w}}\right) \\ &= \frac{z\bar{w}}{|w|^2} \end{aligned}$$

Reciprocal of Complex Number

- If $z = x + iy \neq 0$, then its reciprocal is

$$\begin{aligned} z^{-1} &= \frac{1}{z} \\ &= \frac{\bar{z}}{|z|^2} \\ &= \frac{x - iy}{x^2 + y^2} \end{aligned}$$

Polar Coordinates



- Recall that polar coordinates are given by

$$(x, y) = (r\cos\theta, r\sin\theta) = r(\cos\theta, \sin\theta),$$

where $r = \sqrt{x^2 + y^2}$

- Therefore, each complex number $z = x + iy$ can be written as

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta),$$

where $r = |z|$

Complex Exponential

- Recall that if $x \in \mathbb{R}$, then

$$e^x = \sum_{k=0}^{k=\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

- If $z \in \mathbb{C}$, we can define the exponential of z to be

$$e^z = \sum_{k=0}^{k=\infty} \frac{z^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

- Properties

$$\begin{aligned} e^{z+w} &= e^z e^w \\ e^{kz} &= (e^z)^k \text{ if } k \in \mathbb{Z} \end{aligned}$$

Euler's Formula

- If $z = i\theta$, then

$$
\begin{aligned}
e^{i\theta} &= \sum_{k=0}^{k=\infty} \frac{(i\theta)^k}{k!} \\
&= 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \cdots \\
&= 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \cdots \\
&= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\
&= \sum_{j=0}^{j=\infty} (-1)^j \frac{\theta^{2j}}{(2j)!} + i\sum_{j=0}^{\infty} (-1)^j \frac{\theta^{2j+1}}{(2j+1)!} \\
&= e^{i\theta}
\end{aligned}
$$

- Therefore, we define

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Polar Form of Complex Number

- Any $z \in \mathbb{C}$ can be written as
$$z = re^{i\theta},$$
where $r = |z|$
- Key examples
$$e^{i\frac{\pi}{2}} = i$$
$$e^{i\pi} = -1$$
$$e^{i2\pi} = 1$$
- In general,
$$e^{i\theta} = 1 \iff \theta = 2\pi j \text{ for some } j \in \mathbb{Z}$$
- Therefore, if $z = re^{i\theta} = se^{i\phi}$ is nonzero, then
$$1 = \frac{re^{i\theta}}{se^{i\phi}} = \frac{r}{s}e^{i(\theta-\phi)},$$
which implies $r = s$ and $\theta = \phi + 2\pi j$ for some integer j

Angle Addition Formula

- Observe that by the angle addition formulas for sine and cosine,

$$\begin{aligned} e^{i(\alpha+\beta)} &= \cos(\alpha+\beta) + i\sin(\alpha+\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \\ &= \cos\alpha(\cos\beta + i\sin\beta) + \sin\alpha(-\sin\beta + i\cos\alpha) \\ &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= e^{i\alpha}e^{i\beta} \end{aligned}$$

- Therefore, if $z = re^{i\theta}$, then

$$e^{i\phi}z = e^{i\phi}re^{i\theta} = re^{i(\theta+\pi)}$$

- Multiplying z by $e^{i\phi}$ rotates z counterclockwise by ϕ radians

Roots of a Complex Number

- Let $z = r^{i(\theta+2\pi j)}$ and k be a nonzero integer
- A k-th root of z is a complex number $w = se^{i\phi}$ such that

$$re^{i(\theta+2\pi j)} = z = w^k = (se^{i\phi})^k = s^k e^{ik\phi}$$

- This implies that

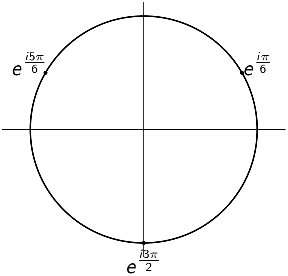
$$r = s^k \text{ and } \theta + 2\pi j = k\phi$$

- It follows that if

$$s = r^{1/k} \text{ and } \phi = \frac{\theta}{k} + 2\pi \left(\frac{j}{k}\right)$$

  then $w^k = z$ and therefore w is a k-th root
- Examples
  - A square root of i is $e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$
  - A cube root of i is $e^{i\frac{\pi}{6}} \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$

Cube Roots of i



- The cube roots of $i = e^{\frac{i\pi}{2}} = e^{\frac{i\pi}{2}+2\pi} = e^{\frac{i\pi}{2}+4\pi}$ are

$$e^{\frac{i\pi}{6}} = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{i}{2}$$

$$e^{\frac{i5\pi}{6}} = \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) = \frac{-\sqrt{3}}{2} + \frac{i}{2}$$

$$e^{\frac{i3\pi}{2}} = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = -i$$

Complex Functions

- A complex function is a function (i.e., map) whose domain is a subset of $\mathbb{C}$ and codomain is $\mathbb{C}$
- If $D \subset \mathbb{C}$, a function $f : D \to \mathbb{C}$ can always be written in terms of two real functions of two variables,

$$f(x+iy) = g(x,y) + ih(x,y)$$

- If a function $f : D \to \mathbb{C}$ can be written as a formula, the formula can be written in terms of x and y or in terms of z and $\bar{z}$
- Examples:

$$f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i2xy$$
$$f(z) = \bar{z} = x - iy$$
$$f(z) = x = \frac{z+\bar{z}}{2}$$

Examples of Complex Functions

- Power function: Given a nonnegative integer n,

$$\begin{aligned} p : \mathbb{C} &\to \mathbb{C} \\ z &\mapsto z^n \end{aligned}$$

- Given a nonnegative integer n,

$$\begin{aligned} \rho : \mathbb{C} &\to \mathbb{C} \\ x + iy &\mapsto x^n + iy^n \end{aligned}$$

- Reciprocal function:

$$\begin{aligned} \rho : \mathbb{C}\backslash\{0\} &\to \mathbb{C} \\ z &\mapsto \frac{1}{z} \end{aligned}$$

- Conjugation:

$$\begin{aligned} \Xi : \mathbb{C} &\to \mathbb{C} \\ z &\mapsto \bar{z} \end{aligned}$$

Cube Root Function

- Observe that the map

$$\begin{aligned} (0,\infty) \times [0, 2\pi) &\to \mathbb{C} \\ (r, \theta) &\mapsto re^{i\theta} \end{aligned}$$

  is bijective

- Therefore, we can define a cube root function to be

$$\begin{aligned} \mathbb{C} &\to \mathbb{C} \\ re^{i\theta} &\mapsto r^{1/3} e^{i\frac{\theta}{e}}, \end{aligned}$$

  where $0 \leq \theta < 2\pi$