MATH-GA2450 Complex Analysis

Complex Differentiability
Cauchy-Riemann Equations

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September 12, 2024

Real Differentiability

- Recall that if $I \subset \mathbb{R}$ is an open interval, a function $f : I \to \mathbb{R}$ is **differentiable** at $x \in I$ if the limit
$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$
exists
- I.e., for any sequence $(x_k : k \geq 0) \subset I \backslash \{x\}$ such that
$$\lim_{k \to \infty} x_k = x,$$
the limit
$$\lim_{k \to \infty} \frac{f(x_k) - f(x)}{x_k - x}$$
exists
- If so, the **derivative** of f at x is defined to be
$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$
- Immediate consequence: If f is differentiable at $x \in I$, then it is continuous at x

Complex Differentiability

- Given an open $U \subset \mathbb{C}$, a function $f : U \to \mathbb{C}$ is **differentiable** at $z \in U$ if
$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$
exists
- If so, the **derivative** of f at z is defined to be the value of the limit and denoted
$$f'(z) \text{ or } \frac{df}{dz}(z)$$
- Immediate consequence: If f is differentiable at z, then it is continuous at z

Basic Properties of Derivatives

- If $f : U \to \mathbb{C}$ is constant, then for any $z \in U$, $f'(z) = 0$
- **Sum rule:** If f and g are differentiable at z, then so is $f + g$ and
$$(f+g)'(z) = f'(z) + g'(z)$$
- **Product rule:** If f and g are differentiable at z, then so is fg and
$$(fg)'(z) = f'(z)g(z) + f(z)g'(z)$$
- **Quotient rule:** If f and g are differentiable at z and $g(z) \neq 0$, then f/g is differentiable at z and
$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$$
- **Chain rule:** If f is differentiable at z and g is differentiable at $f(z)$, then
$$(g \circ f)'(z) = g'(f(z))f'(z)$$

Holomorphic Functions

- A function $f : U \to \mathbb{C}$, where $U \subset \mathbb{C}$ is open, is **differentiable** or **holomorphic** on U if it is differentiable at each $z \in U$
- A holomorphic function $f : U \to V$, where U, V are open, is a **holomorphic isomorphism** if there exists a holomorphic function $g : V \to U$ such that

$$g \circ f = \mathrm{id}_U \text{ and } f \circ g = \mathrm{id}_V$$

Complex Versus Real Differentiability (Part 1)

- A complex function $f : U \to \mathbb{C}$ can be written in terms of real functions u and v in two variables as follows:

$$\forall x + iy \in U,\ f(x + iy) = u(x, y) + iv(x, y)$$

- The complex difference quotient can be written as

$$\begin{aligned}\frac{f(w) - f(z)}{w - z} &= \frac{(u(s,t) + iv(s,t)) - (u(x,y) + iv(x,y))}{(s + it) - (x + iy)} \\ &= \frac{(u(s,t) - u(x,y)) + i(v(s,t) - v(x,y))}{(s - x) + i(t - y)}\end{aligned}$$

- Suppose that $f'(z) = a + ib$
- Therefore, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|w - z| \leq \delta \implies \left| \frac{f(w) - f(z)}{w - z} - (a + ib) \right| \leq \epsilon$$

Complex Versus Real Differentiability (Part 2)

- This is equivalent to

$$\sqrt{(s-x)^2+(t-y)^2} \le \delta$$
$$\implies \left| \frac{(u(s,t)-u(x,y))+i(v(s,t)-v(x,y))}{(s-x)+i(t-y)} - (a+ib) \right| \le \epsilon$$

- In particular, if $t = y$, then this implies that

$$|s-x| \le \delta$$
$$\implies \left| \frac{(u(s,y)-u(x,y))+i(v(s,y)-v(x,y))}{s-x} - (a+ib) \right| \le \epsilon$$
$$\implies \left| \frac{u(s,y)-u(x,y)}{s-x} - a + i\left(\frac{v(s,y)-v(x,y)}{s-x} - b \right) \right| \le \epsilon$$
$$\implies \left| \frac{u(s,y)-u(x,y)}{s-x} - a \right| + \left| \frac{v(s,y)-v(x,y)}{s-x} - b \right| \le \epsilon$$

Complex Versus Real Differentiability (Part 3)

- Therefore, if f is differentiable at $z = x + iy$, then for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|s - x| \leq \delta$$
$$\implies \left| \frac{u(s,y) - u(x,y)}{s - x} - a \right| \leq \epsilon \text{ and } \left| \frac{v(s,y) - v(x,y)}{s - x} - b \right| \leq \epsilon$$

- It follows that

$$\lim_{s \to x} \frac{u(s,y) - u(x,y)}{s - x} = a \text{ and } \lim_{s \to x} \frac{v(s,y) - v(x,y)}{s - x} = b$$

- In other words, if we denote the partial derivatives of u and v with respect to x by by u_x and v_x, then

$$u_x(x,y) = a \text{ and } v_x(x,y) = b$$

Complex Versus Real Differentiability (Part 4)

- The above can be written more briefly as follows:

$$\begin{aligned} f'(z) &= \lim_{w\to z} \frac{f(w)-f(z)}{w-z} \\ &= \lim_{s\to x} \left(\frac{u(s,y)-u(x,y)}{s-x}\right) + i\left(\frac{v(s,y)-v(x,y)}{s-x}\right) \\ &= u_x(x,y) + iv_x(x,y) \end{aligned}$$

- The same calculation with $s = x$ and $t \to y$ gives

$$\begin{aligned} f'(z) &= \lim_{w\to z} \frac{f(w)-f(z)}{w-z} \\ &= \lim_{t\to y} \left(\frac{u(x,t)-u(x,y)}{i(t-y)}\right) + i\left(\frac{v(x,t)-v(x,y)}{i(t-y)}\right) \\ &= v_y(x,y) - iu_x(x,y) \end{aligned}$$

Holomorphic $\implies$ Cauchy-Riemann Equations

- Therefore, if $f = u + iy$ is complex differentiable at $z = x + iy$, then
  - Partial derivatives of u and v at (x, y) exist
  - **AND**

$$u_x = v_y \text{ and } u_y = -v_x$$

- These are called the **Cauchy-Riemann equations**
- Complex differentiability is a much stronger property than real differentiability

Cauchy-Riemann Equations $\implies$ Holomorphic

- Let $O \subset \mathbb{C}$ be open and let

$$\widehat{O} = \{(x, y) \ : \ x + iy \in O\} \subset \mathbb{R}^2$$

- Let $u : \widehat{O} \to \mathbb{R}$ and $v : \widehat{O} \to \mathbb{R}$ be C^1 functions such that

$$u_x = v_y \text{ and } u_y = v_x$$

- Let $f : O \to \mathbb{C}$ be given by

$$f(x + iy) = u(x, y) + iv(x, y)$$

- To prove that f is holomorphic, need to show that for each $z \in O$,

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists

Differential of a 2-Dimensional Map

- Let $\widehat{O} \subset \mathbb{R}^2$ be open and consider a map $F : \widehat{O} \to \mathbb{R}^2$, where
$$F(x, y) = (u(x, y), v(x, y))$$
- Given $(x, y) \in \widehat{O}$ and a matrix
$$M = \begin{bmatrix} a & c \\ b & d \end{bmatrix},$$
let
$$\begin{aligned} E(s, t) &= F(s, t) - F(x, y) - M(s - x, t - y) \\ &= \begin{bmatrix} u(s, t) - u(x, y) \\ v(s, t) - v(x, y) \end{bmatrix} - \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} s - x \\ t - y \end{bmatrix} \end{aligned}$$
- Recall that F is **differentiable** at (x, y) if there exists a matrix M such that
$$\lim_{(s,t)\to(x,y)} \frac{|E(s, t)|}{|(s, t)|} = 0.$$

Jacobian of a Differentiable Map

- If F is differentiable at x, then the matrix M is called the **Jacobian of** F **at** (x, y) and denoted $DF(x, y)$
- Moreover,

$$DF(x, y) = \begin{bmatrix} u_x(x, y) & u_y(x, y) \\ v_x(x, y) & v_y(x, y) \end{bmatrix}$$

Cauchy-Riemann Equations $\implies$ Holomorphic

- Let $F = (u, v) : \widehat{O} \to \mathbb{R}^2$ be a map that is differentiable at $(x, y) \in \widehat{O}$
- Assume that the Cauchy-Riemann equations hold:

$$u_x = v_y \text{ and } u_y = v_x$$

- Let $f(x + iy) = u(x, y) + iv(x, y)$
- Claim: f is complex differentiable at $z = x + iy$

Proof that f is Complex Differentiable at $x + iy$

- The Cauchy-Riemann equations imply that

$$DF(x,y) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

where $a = u_x = v_y$ and $b = v_x = -u_y$

- Therefore,

$$\begin{aligned} E(s,t) &= F(s,t) - F(x,y) - DF(x,y)(s-x, t-y) \\ &= \begin{bmatrix} u(s,t) - u(x,y) - (a(s-x) - b(t-y)) \\ v(s,t) - v(x,y) - (b(s-x) + a(t-y)) \end{bmatrix} \end{aligned}$$

- F differentiable implies

$$\lim_{(s,t)\to(x,y)} \frac{|E(s,t)|}{|(s-x, t-y)|} = 0$$

Proof that f is Complex Differentiable at $x + iy$

- On the other hand,

$$\begin{aligned} & f(w) - f(z) - (a+ib)(w-x) \\ & = u(s,t) - u(x,y) + i(v(s,t) - v(x,y)) \\ & \quad - ((a+ib)(s-x) + i(t-y)) \\ & = u(s,t) - u(x,y) - a(s-x) + b(t-y) \\ & \quad + i(v(s,t) - v(x,y) - b(s-x) - a(t-y)) \end{aligned}$$

- Therefore,

$$|f(w) - f(z) - (a+ib)(w-z)| = |E(s,t)|$$

Proof that f is Complex Differentiable at $x + iy$

- It follows that

$$\begin{aligned}
&\lim_{w\to z}\left|\frac{f(w)-f(z)}{w-z}-(a+ib)\right| \\
&= \lim_{w\to z}\left|\frac{f(w)-f(z)-(a+ib)(w-z)}{w-z}\right| \\
&= \frac{|f(w)-f(z)-(a+ib)(w-z)|}{|w-z|} \\
&= \frac{|E(s,t)|}{|(s-x,t-y)|} \\
&= 0
\end{aligned}$$

- This proves that f is differentiable at z and

$$f'(z) = a + ib = u_x + iv_x = v_x - iu_y$$

Basic Example: Linear Function

- The simplest non-constant function $f : \mathbb{C} \to \mathbb{C}$ is a linear function:
$$f(x+iy) = ax + cy + i(bx + dy)$$
- $f = u + iv$, where
$$u(x,y)ax + cy \text{ and } v(x,y) = bx + dy$$
,
- The partial derivatives of u and v are
$$u_x = a, \ u_y = c, \ v_x = b, \ v_y = d$$
- f is holomorphic if and only if $a = d$ and $c = -b$ and therefore
$$f(x+iy) = ax - by + i(bx + ay) = (a+ib)(x+iy),$$
i.e.,
$$f(z) = \alpha z, \text{ where } \alpha = a + ib \text{ and } z = x + iy$$