MATH-GA2450 Complex Analysis

Polar Form of Piecewise C^1 curve
Winding Number
Residue Theorem for Laurent Series
Cauchy Integral Formula for Analytic Functions

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Parameterized Curve in Polar Form

- **Lemma.** If $c : [a, b] \to \mathbb{C} \backslash \{0\}$ is a piecewise C^1 curve and

  $$c(a) = r_0 e^{i\theta_0},$$

  then there exists unique piecewise C^1 real functions

  $$r : [a, b] \to \mathbb{R} \text{ and } \theta : [a, b] \to \mathbb{R}$$

  such that

  $$c(t) = r(t)e^{i\theta(t)}, \; r(a) = r_0, \, \theta(a) = \theta_0 \tag{1}$$

Polar Form of Piecewise C^1 Curve (Part 1)

- Given $r : [a, b] \to (0, \infty)$ and $\theta : [a, b] \to \mathbb{R}$ be piecewise C^1 functions such that

$$r(a) = r_0 \text{ and } \theta(a) = \theta_0$$

- If

$$\phi(t) = \frac{c(t)}{r(t)e^{i\theta(t)}},$$

then $\phi(a) = 1$ and therefore (1) holds if and only if for each $t \in [a, b]$,

$$r(t) = |\phi(t)| \text{ and } \phi'(t) = 0$$

Polar Form of Piecewise C^1 Curve (Part 2)

- Therefore, $\phi = 1$ if and only if $r = |\phi|$ and

$$\begin{aligned} 0 &= \phi'(t) \\ &= \frac{d}{dt}\left(\frac{c(t)}{re^{i\theta}}\right) \\ &= \frac{c}{re^{i\theta}}\left(\frac{c'}{c} - \frac{r'}{r} - i\theta'\right) \end{aligned}$$

- Since

$$\frac{c}{e^{i\theta}} \neq 0,$$

this holds if and only if

$$r = |c| \text{ and } \frac{c'}{c} = \frac{r'}{r} + i\theta',$$

which holds if and only if for each $t \in [a, b]$,

$$r(t) = |c(t)| \text{ and } \theta(t) = \theta(a) + \frac{1}{i}\int_{s=a}^{s=t} \frac{c'}{c} - \frac{r'}{r}\, dx$$

Polar Form of Piecewise C^1 Curve Centered at z_0

- Given $z_0 \in \mathbb{C}$, the polar form of a piecewise C^1 curve $c : [a, b] \to \mathbb{C}\backslash\{z_0\}$ centered at z_0 consists of functions

$$r : [a, b] \to \mathbb{R} \text{ and } \theta : [a, b] \to \mathbb{R}$$

  such that

$$c(t) = z_0 + r(t)e^{i\theta(t)}$$

- By the lemma, the curve $\tilde{c}(t) = c(t) - z_0$ has a polar form centered at 0,

$$\tilde{c}(t) = r(t)e^{i\theta(t)}$$

- Therefore,

$$c(t) = z_0 + r(t)e^{i\theta(t)}$$

Winding Number of a Closed Curve Around z_0

- Let $z_0 \in \mathbb{C}$ and $c : [a, b] \to \mathbb{C} \backslash \{z_0\}$ be a closed piecewise C^1 curve such that
$$c(t) = z_0 + r(t)e^{i\theta(t)}$$
- The **winding number** of c around z_0 is defined to be
$$W(c, z_0) = \frac{1}{2\pi}(\theta(b) - \theta(a))$$
- Since
$$e^{i\theta(b)} = e^{i\theta(a)},$$
it follows that $\theta(b) - \theta(a)$ is an integer multiple of 2π and therefore $W(c, z_0) \in \mathbb{Z}$
- Contour integral formula:
$$\begin{aligned} \frac{1}{2\pi i}\int_c \frac{dz}{z - z_0} &= \int_{t=a}^{t=b} \frac{c'(t)}{c(t) - z_0}\, dt \\ &= 2\pi(\theta(b) - \theta(a)) \\ &= W(c, z_0) \end{aligned}$$

Winding Number of Star-Shaped Curve is 1

- A closed piecewise C^1 curve $c : [a, b] \to \mathbb{C}\backslash\{z_0\}$ with polar form
$$c(t) = z_0 + r(t)e^{i\theta(t)}$$
is **star-shaped** around z_0 if
$$\forall t \in [a, b],\ \theta'(t) \neq 0$$
and for each $\theta_0 \in [0, 2\pi]$ there exists a unique $t_0 \in [a, b]$ such that
$$c(t_0) = z_0 + r(t_0)e^{i\theta_0}$$
- This implies that θ is either an increasing function or a decreasing function
- It follows that
$$W(c, z_0) = \theta(b) - \theta(a) = \pm 2\pi$$

Residue Theorem For Laurent Series

- **Theorem.** If, for each $z \in D(z_0, R)$,

$$f(z) = \sum_{k=k_0} a_k (z - z_0)^k$$

converges absolutely and $c : [a, b] \to D(z_0, R) \backslash \{z_0\}$ is a closed piecewise C^1 curve, then

$$\frac{1}{2\pi i} \int_c f(z)\, dz = W(c, z_0) a_{-1}$$

- **Proof.**

$$\begin{aligned} \frac{1}{2\pi i} \int_c f(z)\, dz &= \frac{1}{2\pi i} \int_c \sum_{k=k_0} a_k (z - z_0)^k \, dz \\ &= \frac{a_{-1}}{2\pi i} \int_c \frac{dz}{z - z_0} \\ &= a_1 W(c, z_0) \end{aligned}$$

Cauchy Integral Formula for Analytic Function

- **Theorem.** If, for each $z \in D(z_0, R)$,

$$f(z) = \sum_{k=0} a_k (z - z_0)^k$$

converges absolutely and $c : [a, b] \to \mathbb{C} \setminus \{z_0\}$ is a closed piecewise C^1 curve, then

$$\frac{1}{2\pi i} \int_c \frac{f(z)}{z - z_0}\, dz = W(c, z_0) f(z_0)$$