MATH-GA2450 Complex Analysis

Isolated Singularities
Poles
Meromorphic Functions
Esential Singularities

Deane Yang

Courant Institute of Mathematical Sciences
New York University

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Isolated Singularities

- Given $z_0 \in \mathbb{C}$ and $R > 0$, a holomorphic function $f : D(z_0, R) \backslash \{z_0\} \to \mathbb{C}$ is said to have an **isolated singularity** at z_0
- An isolated singularity is **removable** if f can be extended to be a holomorphic function on $D(z_0, R)$
- **Theorem.** If for some $0 < S < R$, f is bounded on $D(z_0, S)$, then z_0 is a removable singularity

Proof (Part 1)

- Assume that for any $z \in D(z_0, S)\backslash\{z_0\}$, $|f(z)| \leq C$
- For any $s > 0$, f has a Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k,$$

where

$$a_k = \frac{1}{2\pi i} \int_{\partial D(z_0,s)} \frac{f(z)}{(z - z_0)^{k+1}} \, dz$$

- Therefore,

$$\begin{aligned} |a_k| &= \frac{1}{2\pi} \left| \int_{\partial D(z_0,s)} \frac{f(z)}{(z - z_0)^{k+1}} \, dz \right| \\ &\leq \frac{1}{2\pi} \int_{\partial D(z_0,s)} |f(z)||z - z_0|^{-k-1} \, dz \\ &= Cs^{-k-1} \end{aligned}$$

Proof (Part 2)

- If $k \leq -1$, then $p = -k - 1 > 0$
- Since $|a_k| < Cs^p$ for any $s > 0$, it follows that $a_k = 0$
- It follows that

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

Poles

- If $f : D(z_0, r)\backslash\{0\} \to \mathbb{C}$ and its Laurent series has only finitely many negative terms,

$$f(z) = \sum_{k=-n}^{\infty} a_k (z - z_0)^k$$

  where $a_{-n} \neq 0$, then f is said to have a **pole of order** n at z_0
- Equivalently, $f : D(z_0, r)\backslash\{z_0\}$ has a pole of order n at z_0 if and only if the function $h(z) = (z - z_0)^n f(z)$ has a removable singularity at z_0 and $h(z_0) \neq 0$
- A pole of order 1 is also called a **simple pole**

Meromorphic Functions

- Let $O \subset \mathbb{C}$ be open and $z_1, \ldots, z_N \in O$
- A holomorphic function $f : O \backslash \{z_1, \ldots, z_N\} \to \mathbb{C}$ is **meromorphic on** O if each z_k is a pole

Essential Singularities

- z_0 is an **essential singularity** of $f : D(z_0, r) \to \mathbb{C}$ if the Laurent series of f has infinitely many negative terms
- Example

$$e^{1/z} = \sum_{k=-\infty}^{0} \frac{z^k}{(-k)!}$$