MATH-GA2450 Complex Analysis

Residue Formula
Order of Meromorphic Function at Point

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Residue of a Meromorphic Function

- Let
$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$
where the sum converges absolutely on $D(z_0, R) \backslash \{z_0\}$ for some $R > 0$
- Recall that if $c = \partial D(z_0, r)$, oriented counterclockwise, then
$$\frac{1}{2\pi i} \int_c (z - z_0)^n \, dz = \begin{cases} 1 & \text{if } n = -1 \\ 0 & \text{if } n \neq -1 \end{cases}$$
- Therefore, if $0 < r < R$, then
$$\begin{aligned} \int_c f(z) \, dz &= \int_c \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \, dz \\ &= \sum_{n=-\infty}^{\infty} a_n \int_c (z - z_0)^n \, dz \\ &= 2\pi i a_{-1} \end{aligned}$$

Residue Formula

- If f is a homorphic function on $D(z_0, r)\backslash\{z_0\}$ for some $r > 0$ and its Lauren series is

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k,$$

then we define the **residue** of f at z_0 to be

$$\mathrm{Res}_{z_0} f = a_{-1}$$

- Let $O \subset \mathbb{C}$ be open and $c \subset O$ be a closed chain that is null homologous in O
- If f is a holomorphic function on $O\backslash\{z_1, \ldots, z_N\}$, then

$$\int_c f(z)\, dz = 2\pi i \sum_{k=1}^{N} W(c, z_k)\, \mathrm{Res}_{z_k} f$$

Proof of Residue Formula (Part 1)

- For each $k = 1, \ldots, N$, let $D(z_k, r_k) \subset O$ be disks such that for each $1 \leq j, k \leq N$,

$$\overline{D(z_k, r_k)} \cap \overline{D(z_k, r_k)} = \emptyset$$

- Therefore, if $c_k = \partial D(z_k, r_k)$ oriented counterclockwise, then

$$W(c_k, z_k) = 1 \text{ and } W(c_k, z_j) = 0 \text{ if } j \neq k$$

- By definition of the winding number of a chain,

$$\begin{aligned} W\left(c - \sum_{j=1}^{N} W(c, z_j) c_j, z_k\right) &= W(c, z_k) - \sum_{j=1}^{N} W(c, z_j) W(c_j, z_k) \\ &= 0 \end{aligned}$$

Proof of Residue Formula (Part 2)

- Therefore,

$$\int_c f(z)\,dz = 2\pi i \sum_{k=1}^{N} W(c, z_k) \int_{c_k} f(z)\,dz$$
$$= 2\pi i \sum_{k=1}^{N} W(c, z_k) \operatorname{Res}_{z_k} f$$

Residue Using Cauchy Integral Formula

- Recall that if f is holomorphic on an open set containing $\overline{D(z_0, r)}$, then

$$\int_{\partial D(z_0,r)} \frac{f(z)}{(z-z_0)^{k+1}}\, dz = 2\pi i \frac{f^{(k)}(z_0)}{k!}$$

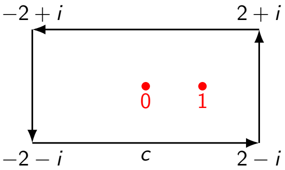
- Suppose

$$h(z) = \frac{f(z)}{(z-z_0)^{k+1}}$$

- The residue of h at z_0 is

$$\begin{aligned} \operatorname{Res}_{z_0} h &= \frac{1}{2\pi i} \int_{\partial D(z_0,r)} h(z)\, dz \\ &= \frac{1}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(z)}{(z-z_0)^{k+1}}\, dz \\ &= \frac{1}{2\pi i} \frac{f^{(k)}(z_0)}{k!} \end{aligned}$$

Example (Part 1)



- Let c be the contour shown above and R the open set inside c
- Consider $\int_c \frac{e^z}{z(z-1)^2}\,dz$
- If

$$f(z) = \frac{e^z}{z(z-1)^2},$$

then

$$\int_c \frac{e^z}{z(z-1)^2}\,dz = 2\pi i(\mathrm{Res}_0 f + \mathrm{Res}_1 f)$$

Example (Part 2)

- If

$$f_1(z) = \frac{e^z}{(z-1)^2}$$

then f_1 is holomorphic on $R\backslash\{-1\}$ and

$$f(z) = \frac{f_1(z)}{z}$$

- Therefore, if $c_1 = \partial D(0, \frac{1}{2})$, then

$$\begin{aligned}
\operatorname{Res}_0 f &= \frac{1}{2\pi i} \int_{c_1} \frac{f_1(z)}{z}\, dz \\
&= f_1(0) \\
&= \frac{1}{(-1)^2} \\
&= 1
\end{aligned}$$

Example (Part 3)

- Similarly, if

$$f_2(z) = \frac{e^z}{z},$$

and $c_2 = \partial D(1, \frac{1}{2})$, then

$$\begin{aligned} \text{Res}_1 f &= \frac{1}{2\pi i} \int_{c_2} \frac{f_2(z)}{(z-1)^2}\, dz \\ &= f_2'(1) \end{aligned}$$

- On the other hand,

$$f_2'(z) = \frac{e^z}{z} - \frac{e^z}{z^2}$$

and therefore

$$\text{Res}_1 = 0$$

- It follows that

$$\int_c \frac{e^z}{z(z-1)^2}\, dz = 2\pi i$$

Order of Meromorphic Function at a Point

- If f is holomorphic on $D(z_0, r)\backslash\{z_0\}$ and has a Laurent series of the form
$$f(z) = \sum_{k=n}^{\infty} a_k (z - z_0)^k,$$
where $a_n \neq 0$, then the **order of f at z_0** is defined to be
$$\text{ord}_{z_0} = n$$
- Observe that
$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{\sum_{k=n}^{\infty} k a_k (z - z_0)^{k-1}}{\sum_{k=n}^{\infty} a_k (z - z_0)^k} \\ &= \left(\frac{1}{z - z_0}\right)\left(\frac{a_n (z - z_0)^{n-1} \sum_{k=n}^{\infty} k a_k (z - z_0)^{k-1}}{a_n (z - z_0)^{k-n} \sum_{k=n}^{\infty} a_k (z - z_0)^{k-n}}\right) \\ &= \left(\frac{1}{z - z_0}\right)\left(\frac{n + \sum_{k=n+1} k a_k (z - z_0)^{k-n}}{1 + \sum_{k=n+1} a_k (z - z_0)^{k-n}}\right)\end{aligned}$$

Orders of Meromorphic Function Inside Closed Chain

- Therefore,
$$\mathrm{Res}_{z_0} \frac{f'}{f} = \mathrm{ord}_{z_0} f$$
- Observe that if $f(z_0) \neq 0$, then $\mathrm{ord}_{z_0} = 0$
- It follows that if $O \subset \mathbb{C}$ is open and f is a meromorphic function with zeros and poles of finite order at $z_1, \ldots, z_N$, then given any closed chain $c \subset O \backslash \{z_1, \ldots, z_N\}$,
$$\int_c \frac{f'(z)}{f(z)}\, dz = W(c, z_1)\, \mathrm{ord}_{z_1} f + \cdots + W(c, z_N)\, \mathrm{ord}_{z_N} f$$