MATH-GA2450 Complex Analysis

Maximum Modulus Principle
Schwarz Lemma

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Local Maximum Modulus Principle

- **Theorem.** If f is holomorphic on an open $O \subset \mathbb{C}$ and $z_0 \in U$ is a maxumum point of $|f(z)|$, i.e, for all $z \in U$,

$$|f(z)| \geq |f(z_0|,$$

then there exists $R > 0$ such that f is constant on $D(z_0, R) \subset O$

Proof (Part 1)

- There exists $R > 0$ such that $D(z_0, R) \subset O$
- For each $0 < r < R$,

$$
\begin{aligned}
|f(z_0| &= \left| \frac{1}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(z)}{z - z_0}\, dx \right| \\
&\leq \frac{1}{2\pi} \int_{t=0}^{t=2\pi} |f(z_0 + re^{it})|\, dt \\
&\leq \frac{1}{2\pi} \int_{t=0}^{t=2\pi} |f(z_0)|\, dt \\
&= |f(z_0)|
\end{aligned}
$$

- It follows that

$$\int_{t=0}^{t=2\pi} |f(z_0)| - |f(z_0 + re^{i\theta}|\, dt = 0$$

- Since the integrand is nonnegative, it is always zero, i.e., for any $0 < r < R$ and $0 \leq t \leq 2\pi$

$$|f(z_0 + re^{i\theta})| = |f(z_0)|$$

Proof (Part 2)

- Therefore, for any $z \in D(z_0, R)$,

$$|f(z)| = |f(z_0)|$$

- If $f = u + iv$, this implies that $u^2 + v^2$ is constant
- Differentiating this, we get

$$uu_x + vv_x = 0 \text{ and } uu_y + vv_y = 0$$

- By Cauchy-Riemann equations,

$$uu_x - vu_y = 0 \text{ and } uu_y + vu_x = 0$$

- This implies

$$(u^2 + v^2)u_x = 0 \text{ and } (u^2 + v^2)u_y = 0$$

Proof (Part 3)

- Since $u^2 + v^2$ is constant, either $u^2 + v^2 = 0$ which implies $u = 0$ or
$$u_x = u_y = 0$$
and therefore u is constant
- Similarly, v is constant
- Therefore, f is constant on $D(z_0, R)$

Global Maximum Modulus Principle

- Let $U \subset \mathbb{C}$ be a connected open set
- Let $f : \overline{U} \to \mathbb{C}$ be a continuous function that is holomorphic on U
- If there exists $z_0 \in \overline{U}$ such that for all $z \in U$,

$$|f(z)| \leq |f(z_0)|,$$

  then $z_0 \in \partial U$
- **Proof.**
  - If not, then $z_0 \in U$
  - By the local maximum modulus principle, there exists $R > 0$ such that f is constant on $D(z_0, R)$
  - Since U is connected, this implies that f is constant on U

Schwarz Lemma (Part 1)

- Let $D = D(0,1)$
- **Theorem.** If $f : D \to D$ is analytic and $f(0) = 0$, then
  - $|f(z)| \leq |z|$ for all $z \in D$
  - If there exists $z_0 \in D \setminus \{0\}$ such that $|f(z_0)| = |z_0|$, then there exists $\phi \in \mathbb{R}$, such that

    $$f(z) = e^{i\phi} z,$$

    i.e., f is a rotation
- **Theorem.** If $f : D \to D$ is analytic and $f(0) = 0$, then
  - $|f'(0)| \leq 1$
  - If $|f'(0)| = 1$, then there exists $\phi \in \mathbb{R}$ such that

    $$f(z) = e^{i\phi} z$$

Proof (Part 1)

- Since $f(0) = 0$,

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty} a_k z^k \\ &= z \sum_{k=0} a_{k+1} z^k \\ &= zg(z), \end{aligned}$$

  where g is holomorphic on D
- Since $|f(z)| < 1$ on D, if $0 < r < 1$, then

$$|z| = r \implies |g(z)| = \frac{|f(z)|}{|z|} < \frac{1}{r}$$

- By the maximum modulus principle, if $z \in \overline{D(0, r)}$,

$$|g(z)| \leq \frac{1}{r}$$

Proof (Part 2)

- It follows that if $z \in D$, then

$$|g(z)| \leq \lim_{r \to 1} \frac{1}{r} = 1$$

and therefore

$$|f(z)| = |g(z)||z| \leq |z|$$

- If there exists $z_0 \in D \backslash \{0\}$ such that

$$|f(z_0)| = |z_0|,$$

then for any $z \in D$,

$$|g(z)| \leq 1 = |g(z_0)|$$

- By the maximum modulus principle, g is constant and therefore

$$f(z) = e^{i\theta} z$$

Proof (Part 3)

- It also follows that

$$|f'(0)| = |a_1| = |g(0)| \leq 1$$

- If $|f'(0)| = 1$, then, since for all $z \in D$, $|g(z)| \leq 1 = |g(0)|$, it follows by the maximum modulus principle that g is constant and therefore

$$f(z) = a_1 z$$