MATH-UA 123 Calculus 3: Critical Points, Optimization

Deane Yang

Courant Institute of Mathematical Sciences
New York University

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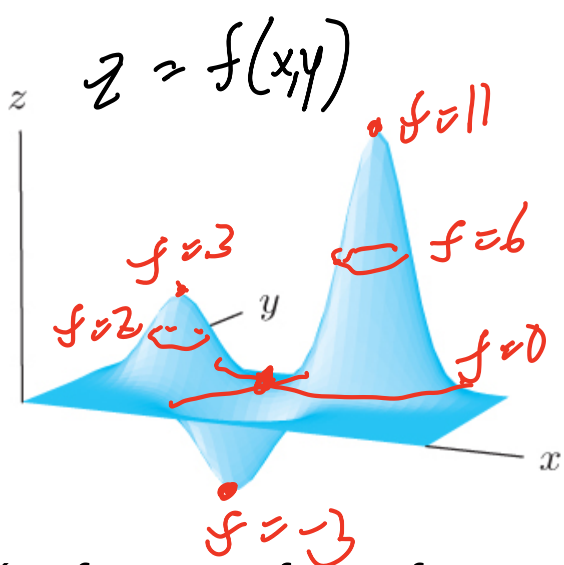
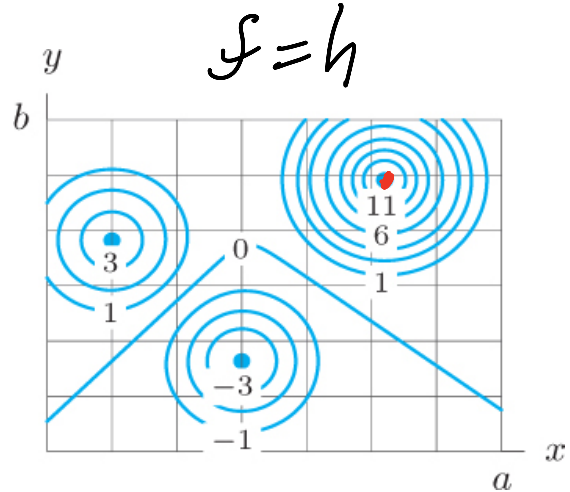
Shape Versus Contours of a Graph

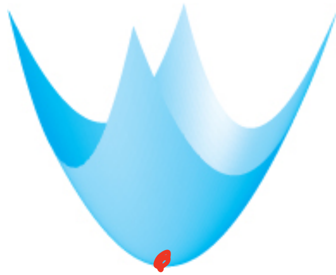




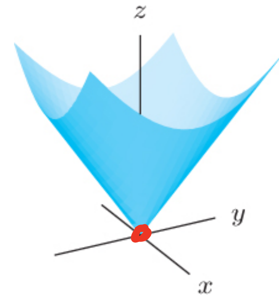
- Key features of a surface:
  - Peaks
  - Bottoms
  - Ridges and valleys between peaks or bottoms
- There are at least 4 points where the gradient is zero
  - Two peaks
  - One bottom
  - One point in between the peaks and bottom, where the contour consists of two intersecting curves

Some Possible Shapes of a Graph

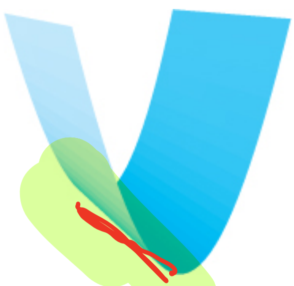
Isolated local maximum



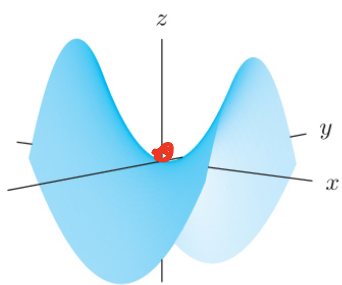
Isolated local minimum



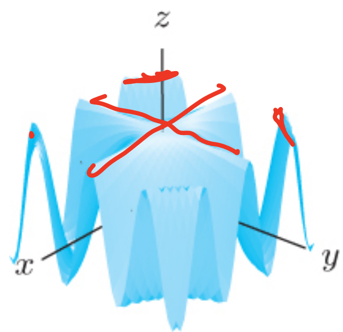
Isolated local minimum



Line of local minima



Saddle point



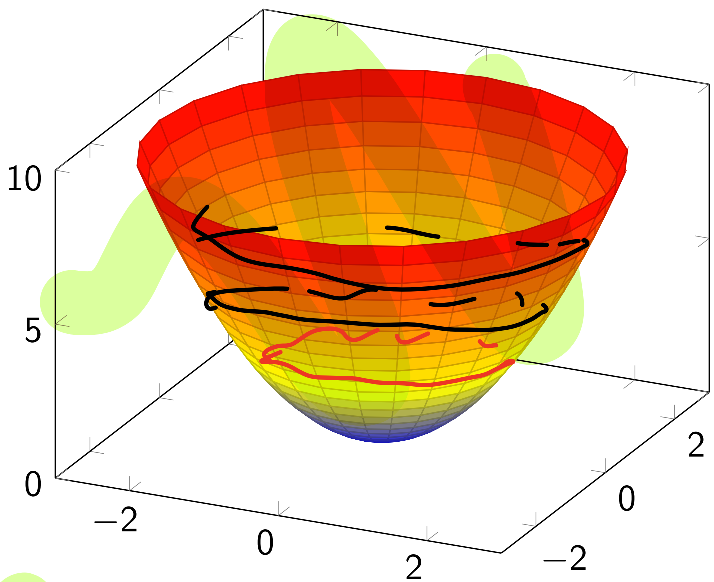
Complicated set of local maxima

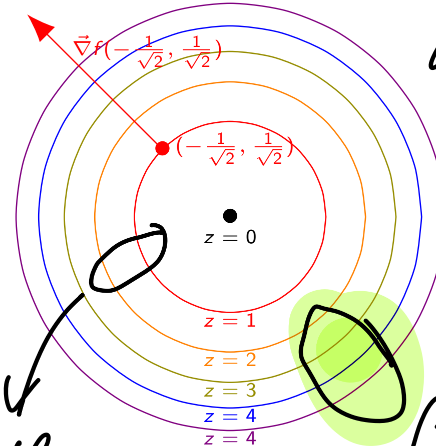
Critical Point of Function

- A point (x_0, y_0) is a critical point of a function f if
  - It is in the domain of f
  - The gradient $\vec{\nabla} f(x_0, y_0, z_0)$ is either zero or undefined
- Possible shapes of a surface near a critical point
  - Isolated local maximum: Top of a hill
  - Isolated local minimum: Bottom of a bowl
  - Curve of local minima: Bottom of a valley
  - Curve of local maxima: Top of a ridge
  - Saddle point
  - Other

Circular Paraboloid

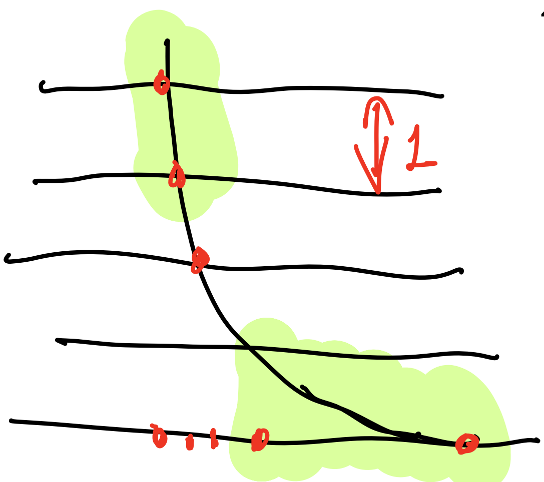
evenly spaced values of z





surface less steep contours widely spaced

surface steeper contours closely spaced.



$$f(x, y) = x^2 + y^2$$

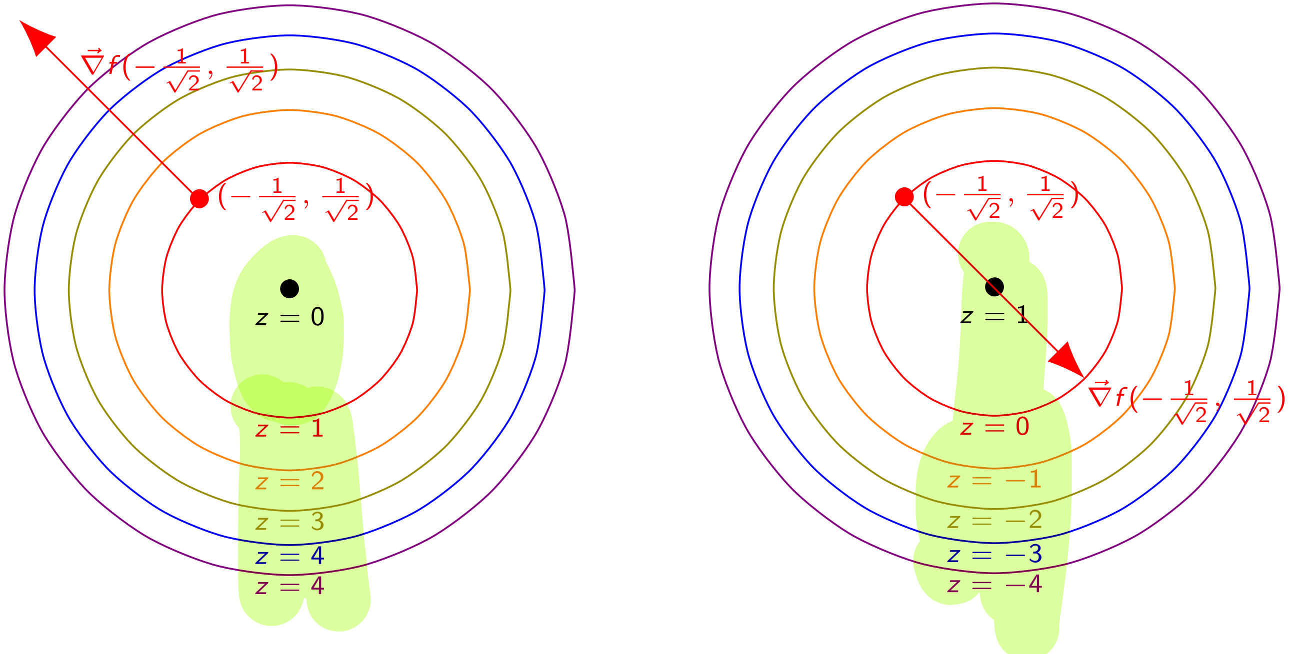
$$\vec{\nabla} f(x, y) = 2\langle x, y\rangle$$

Local minimum

$z = -x^2 - y^2$

Same contour curves as above

Contours of Circular Paraboloids



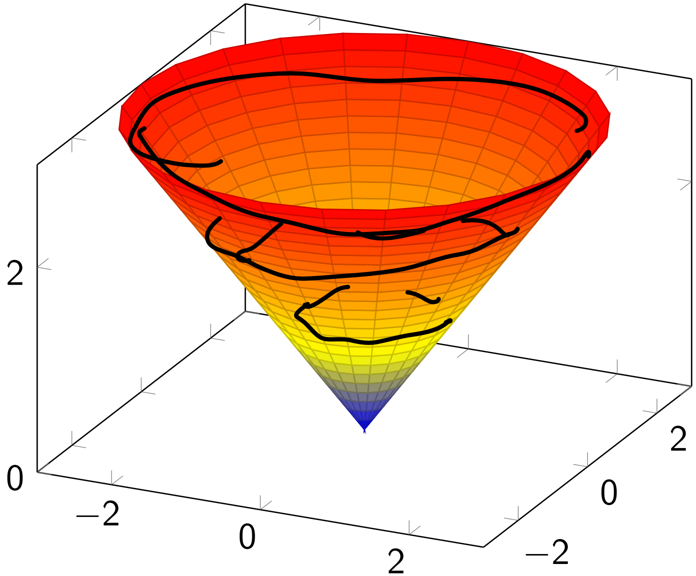
$$f(x, y) = x^2 + y^2$$
$$\vec{\nabla} f(x, y) = 2\langle x, y\rangle$$

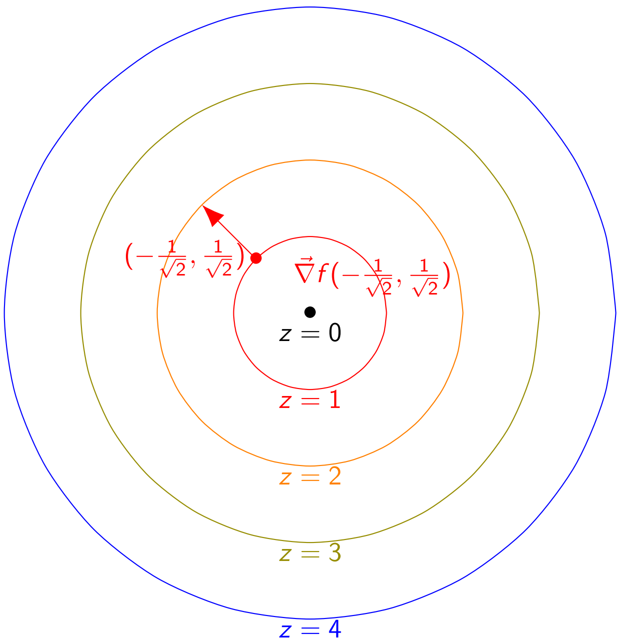
Local minimum

$$f(x, y) = 1 - x^2 - y^2$$
$$\vec{\nabla} f(x, y) = -2\langle x, y\rangle$$

Local maximum

Contour of Circular Cone





$$f(x, y) = \sqrt{x^2 + y^2}$$

$$\vec{\nabla} f = \frac{\langle x, y \rangle}{\sqrt{x^2 + y^2}}$$

Local minimum

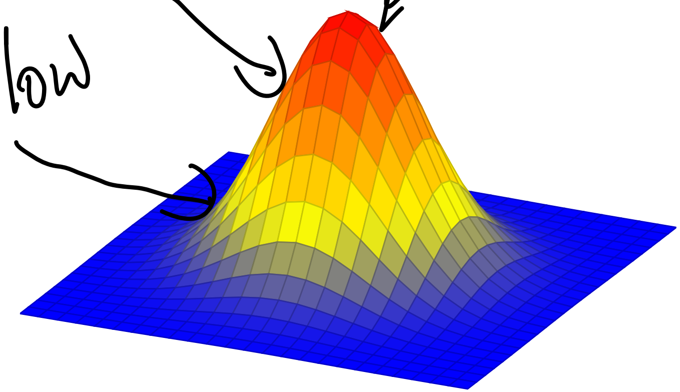
surface always has "same slope"

Another Example of Local Maximum

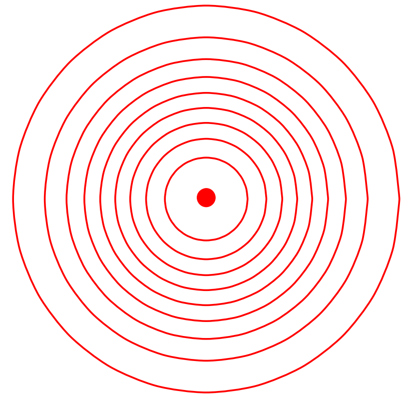
steeper

shallow

shallow



$$z = e^{-x^2-y^2}$$



$$x^2 + y^2 = -\ln(h)$$

Saddle Point

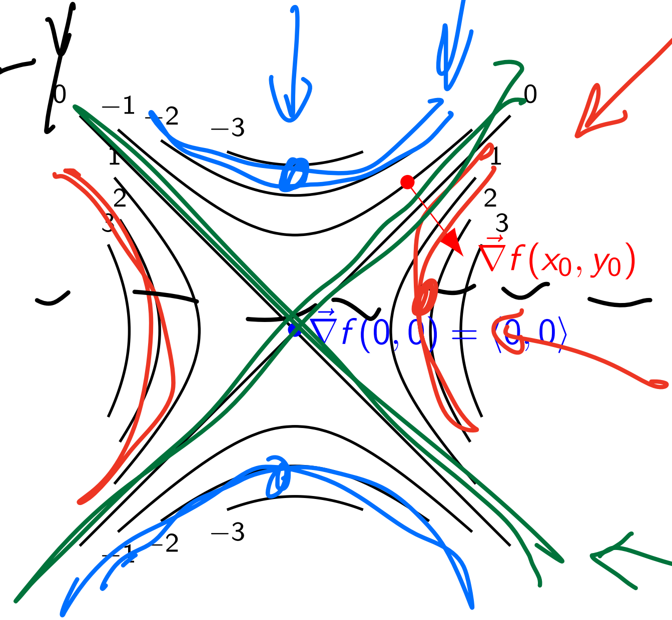
$(0, \sqrt{-h})$ $h < 0$

$h > 0$

$f(x,y) = x^2 - y^2$

$f = h$

$x^2 - y^2 = h$



$\to x$

$(\sqrt{h}, 0)$

$h = 0$

- A point (x_0, y_0) is a saddle point of the graph $z = f(x,y)$, if the contours of f near (x_0, y_0) look like the above
- Examples
  - $(0,0)$ for the function $f(x,y) = x^2 - y^2$
  - $(0,0)$ for the function $f(x,y) = xy$

$x^2 - y^2 = 0$

$(x+y)(x-y) = 0$

$y = \pm x$

Critical Point of Function

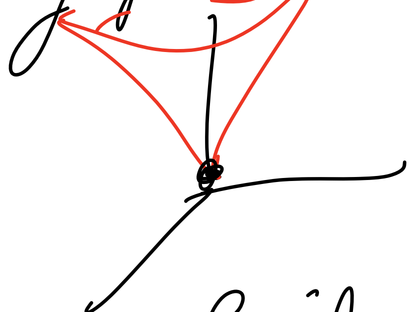
- A critical point of a function f is a point in the domain of f where $\vec{\nabla} f$ is either undefined or equal to the zero vector
- Examples
  - $f(x,y)=\sqrt{x^2+y^2}$: $\vec{\nabla} f(x,y)=\left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$ is undefined at $(0,0)$
  - $f(x,y)=e^{-x^2-y^2}$: $\vec{\nabla} f(x,y)=-2e^{-x^2-y^2}\langle x,y,\rangle$ is the zero vector at $(0,0)$
  - $f(x,y)=x^2-y^2$: $\vec{\nabla} f(x,y)=2\langle x,-y,\rangle$ is the zero vector at $(0,0)$
- If $\vec{\nabla} f(x_0,y_0)=\langle 0,0\rangle$, then the tangent plane

$$z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)=f(x_0,y_0)$$

is horizontal

Examples of critical points

$f(x,y) = \sqrt{x^2+y^2}$: graph is cone



$f_x = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(2x)$

$= \dfrac{x}{\sqrt{x^2+y^2}}$

$f_y = \dfrac{y}{\sqrt{x^2+y^2}}$

$\left.\begin{matrix} f_x \\ f_y \end{matrix}\right\}$ undefined at $(0,0)$

$\Longrightarrow (0,0)$ is only critical point

$f(x,y) = (x-2)^2 - (y+3)^2$

$f_x = 2(x-2)$

$f_y = 2(y+3)$

$\left.\begin{matrix} f_x \\ f_y \end{matrix}\right\} = 0$ if and only if $x=2$ and $y=-3$

$\Longrightarrow (2,-3)$ only critical point.

Types of Critical Points

- Suppose (x_0, y_0) is a critical point of a function $f(x, y)$
- A point (x_0, y_0) is a **local maximum**, if

$$f(x, y) \leq f(x_0, y_0) \text{ for all } (x, y) \text{ near } (x_0, y_0)$$

  - Example: $(0, 0)$ for $f(x, y) = -x^2 - y^2$
- A point (x_0, y_0) is a **local minimum**, if

$$f(x, y) \geq f(x_0, y_0) \text{ for all } (x, y) \text{ near } (x_0, y_0)$$

  - Example: $(0, 0)$ for $f(x, y) = x^2 + y^2$
- A point (x_0, y_0) is a **saddle point**, if it meets the criteria in previous slide
  - Example: $(0, 0)$ for $f(x, y) = x^2 - y^2$
- There are other types of critical points that we will not study
  - Example: $(0, 0)$ for $f(x, y) = xy(x^2 - y^2)$

Tests for Critical Point Type

$$f(x,y) = (x-2)^2 + (y+3)^2$$

- Analyze formula of function
- Draw graph
- Draw contours
- Second derivative test

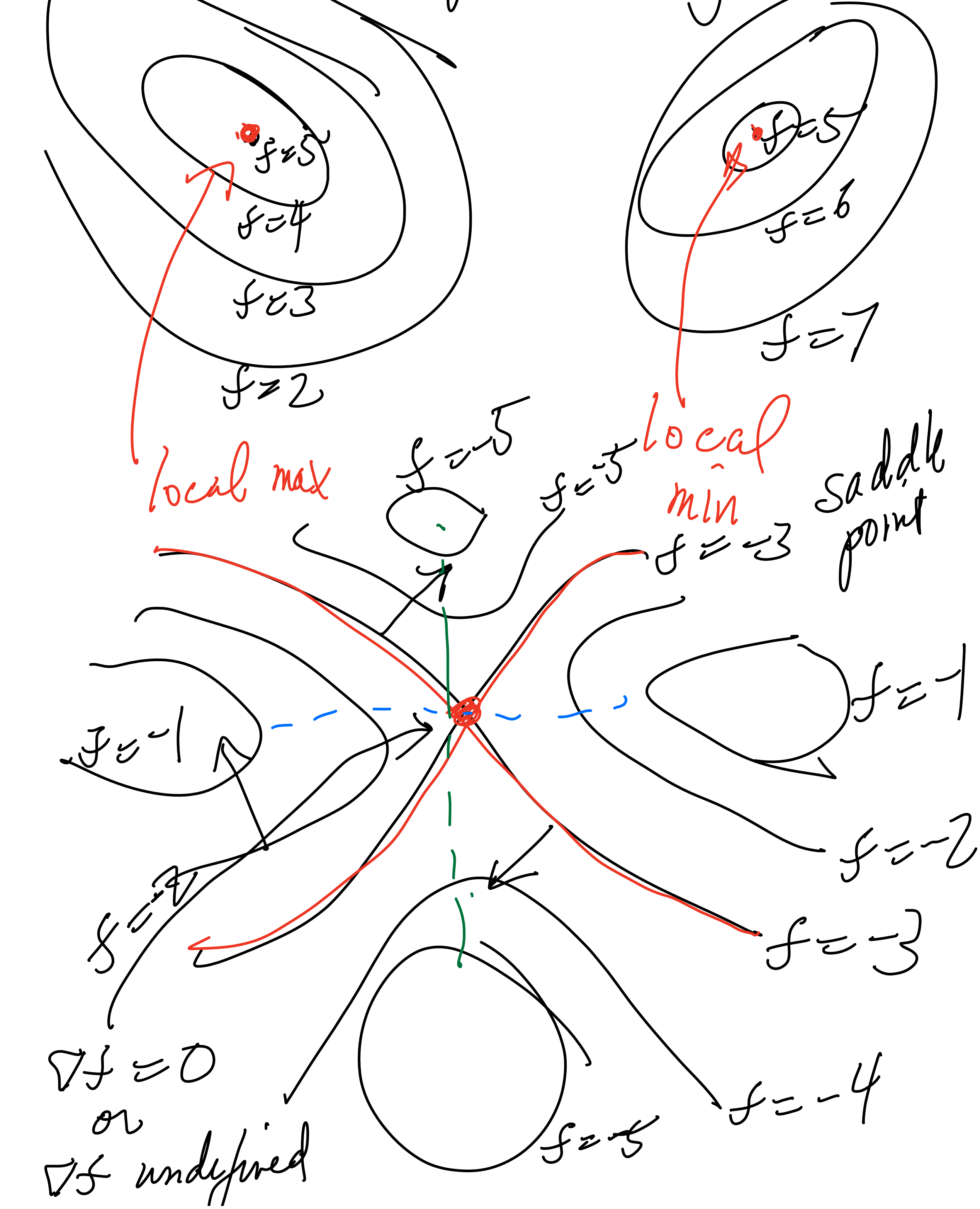
$$f(2,-3) = 0$$

$$f(x,y) > 0$$

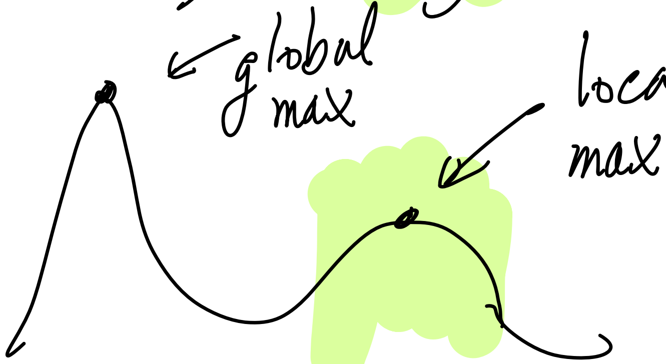
everywhere else

$$\Rightarrow \quad f(x,y) \geq f(0,0), \quad (x,y) \neq (0,0)$$

Critical point using contours



$\Rightarrow$ (0,0) is a global minimum point



global minimum value is 0

min point is (2, −3)

min value is 0

Identify type of critical point

1) Directly from formula
2) Graph
3) Contours
4) 2nd derivative test

Second Derivative Test For Function of One Variable

Suppose x_0 is a critical point of a function $f(x)$, where $f'(x_0) = 0$ and $f''(x_0)$ is defined

- $f''(x_0) > 0 \implies$ local minimum
- $f''(x_0) < 0 \implies$ local maximum
- $f''(x_0) = 0 \implies$ inconclusive

Hessian of Function of Two Variables

- The Hessian of a function $f(x, y)$ at (x_0, y_0) is the matrix

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = \begin{bmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{bmatrix}$$

- H is a matrix of numbers. There should be no x or y in the formula for H
- The determinant of H is defined to be

$$\det H = H_{11}H_{22} - H_{12}H_{21}$$

2^{nd} derivative test for $f(x,y)$

2^{nd} derivatives form 2-by-2 matrix

partial

called Hessian of f

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$$

$H(x,y)$

(remember $\underbrace{(f_x)_y = (f_y)_x}_{f_{xy} = f_{yx}}$)

Suppose (x_0, y_0) is a critical point where $\vec{\nabla} f(x_0, y_0) = 0$

Look at $\underbrace{H(x_0, y_0)}$

has no x's or y's in it because you've replaced x by x_0 and y by y_0

Basic examples

1) $f(x,y) = x^2+y^2$, $f_x = 2x$, $f_y = 2y$

$\Downarrow$ only critical point is $(0,0)$

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Critical point is $(0,0)$

$$H(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$ local min

2) $f(x,y) = 5 - x^2 - y^2$

$f_x = -2x, \quad f_y = -2y$

$$H_{(0,0)} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

only critical point is $(0,0)$

local max

3) $f(x,y) = x^2 - y^2$

$f_x = 2x, \quad f_y = -2y$

$$H = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

saddle point

Second Derivative Test for Function of Two Variables

- The second derivative test of a function $f(x, y)$ at a critical point (x_0, y_0), where $\vec{\nabla} f(x_0, y_0) = \langle 0, 0 \rangle$ and the Hessian is defined
  - If $\det H(x_0, y_0) = 0$, then the test is inconclusive
    - The shape of the surface near (x_0, y_0) can be simple or complicated
    - Look at contours to learn more
  - If $\det H(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point
  - If $\det H(x_0, y_0) > 0$, then there are two possibilities
    - If $H_{11}(x_0, y_0) > 0$ (or, equivalently, $H_{22}(x_0, y_0) > 0$), then (x_0, y_0) is a local minimum
    - If $H_{11}(x_0, y_0) < 0$ (or, equivalently, $H_{22}(x_0, y_0) < 0$), then (x_0, y_0) is a local maximum

If $\vec{\nabla} f(x_0, y_0) = \langle f_x, f_y \rangle$
$= \vec{0}$

(same as $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$)

and $H(x_0, y_0)$ be Hessian at (x_0, y_0)

$$H(x_0, y_0) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

1) Look at $\boxed{\det H = f_{xx} f_{yy} - f_{xy}^2}$

2) If $\det H < 0$, then saddle point

3) If $\det H > 0$,
then either local max
or local min

like Calc 1 2nd deriv test

a) If $f_{xx} > 0$, then
local min

b) If $f_{xx} < 0$, then
local max

or do this using f_{yy}
instead of f_{xx}

Suppose $\det H = \underbrace{f_{xx} f_{yy}}_{\text{positive}} \underbrace{- f_{xy}^2}_{\text{negative}} > 0$

$\Downarrow$

f_{xx}, f_{yy} same sign

Basic Examples

- $f(x,y) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + c$, where $a, b \neq 0$
  - $\vec{\nabla} f = 2\left\langle a^{-2}x, b^{-2}y \right\rangle$
  - Only one critical point: $(0,0)$
  - $H = 2\begin{bmatrix} a^{-2} & 0 \\ 0 & b^{-2} \end{bmatrix}$
  - $\det H(0,0) = 4a^{-2}b^{-2} > 0$ and $H_{11} = 2a^{-2} > 0$
  - $(0,0)$ **is a local minimum**
- $f(x,y) = -\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + c$, where $a, b \neq 0$
  - $\vec{\nabla} f = -2\left\langle a^{-2}x, b^{-2}y \right\rangle$
  - Only one critical point: $(0,0)$
  - $H = -2\begin{bmatrix} a^{-2} & 0 \\ 0 & b^{-2} \end{bmatrix}$
  - $\det H(0,0) = 4a^{-2}b^{-2} > 0$ and $H_{11} = -2a^{-2} > 0$
  - $(0,0)$ **is a local maximum**

$$f(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

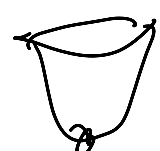
$$f_x=\frac{2x}{a^2}\ ,\ f_y=\frac{2y}{b^2}\Rightarrow(0,0)$$

only critical point

$$\begin{array}{cc} H_{11} & H_{12} \\ \shortparallel & \shortparallel \end{array}$$

$$H=\begin{bmatrix}\frac{2}{a^2} & 0 \\ 0 & \frac{2}{b^2}\end{bmatrix}$$

$$\det H=\left(\frac{2}{a^2}\right)\left(\frac{2}{b^2}\right)>0$$



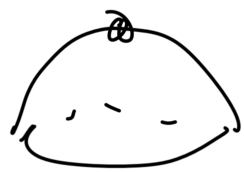
$$f_{xx}=H_{11}>0\Longrightarrow \text{local min}$$

$$f(x,y)=-\frac{x^2}{a^2}-\frac{y^2}{b^2}+c$$

$$H(0,0)=\begin{bmatrix}-\frac{2}{a^2} & 0 \\ 0 & -\frac{2}{b^2}\end{bmatrix}$$

$$\det H = \left(\frac{-2}{a^2}\right)\left(\frac{-2}{b^2}\right) > 0$$

$$f_{xx} = \frac{-2}{a^2} < 0$$

$\Rightarrow$ local max 

$$f(x,y) = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

$$H(0,0) = \begin{bmatrix} \frac{2}{a^2} & 0 \\ 0 & \frac{-2}{b^2} \end{bmatrix}$$

$$\det H = \left(\frac{2}{a^2}\right)\left(\frac{-2}{b^2}\right) < 0$$

Basic Examples

- $f(x,y) = -\frac{x^2}{a^2} + \frac{y^2}{b^2} + c$, where $a, b \neq 0$
  - $\vec{\nabla} f = -2\langle a^{-2}x, b^{-2}y\rangle$
  - Only one critical point: $(0,0)$
  - $H = 2\begin{bmatrix} -a^{-2} & 0 \\ 0 & b^{-2} \end{bmatrix}$
  - $\det H(0,0) = -4a^{-2}b^{-2} < 0$
  - $(0,0)$ **is a saddle point**
- $f(x,y) = axy + c$, where $a \neq 0$
  - $\vec{\nabla} f = a\langle y, x\rangle$
  - Only one critical point: $(0,0)$
  - $H = 2\begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$
  - $\det H(0,0) = -4a^2 < 0$
  - $(0,0)$ **is a saddle point**

Examples Where Second Derivative Test Fails

- $f(x, y) = x^4 + y^4$
  - $\vec{\nabla} f = 4\langle x^3, y^3 \rangle$
  - Only one critical point: $(0, 0)$
  - $H = 4 \begin{bmatrix} x^3 & 0 \\ 0 & y^3 \end{bmatrix}$
  - $\det H(0, 0) = 0$
  - Contours and formula show that $(0, 0)$ is a local minimum
- $f(x, y) = (ax + by)^2 + c$, where $ab \neq 0$
  - $\vec{\nabla} f = 2(ax + by)\langle a, b \rangle$
  - All points (x, y), where $ax + by = 0$ are critical points
  - $H = 2 \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$
  - $\det H(0, 0) = a^2 b^2 - (ab)^2 = 0$
  - Contours and formula show that surface is a parabolic cylinder and all points in the line $ax + by = 0$ are local minimua

$$f(x,y) = x^2y - xy^2$$

$$f_x = 2xy - y^2 = y(2x-y)$$

$$f_y = x^2 - 2xy = x(x-2y)$$

$f_x = 0 \Rightarrow$

$y=0$ or $y=2x$

$y=0$: $f_y = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$

$y=2x$: $f_y = x(x-4x) = -3x^2$

$f_y = 0 \Rightarrow x = 0$

$\Rightarrow y = 2x = 0$

$\Rightarrow (0,0)$ is only critical point

$$H = \begin{bmatrix} 2y & 2x-2y \\ 2x-2y & -2x \end{bmatrix}$$

$f_x = 2xy - y^2$

$f_y = x^2 - 2xy$

$$H(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(Forgotten possibility: $\det H = 0$

Inconclusive

Try other methods)

Example: $f(x,y) = x^4 + y^4$

$f_x = 4x^3$, $f_y = 4y^3 \Rightarrow$ only critical point is $(0,0)$

$$H = \begin{bmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{bmatrix}$$

$$\Rightarrow H(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2nd derivative test fails

$f(0,0) = 0$ and $f(x,y) > 0$
for all other (x,y)

$\Rightarrow$ $(0,0)$ is a minimum point

$$f(x,y) = (2x+3y+4)^2$$

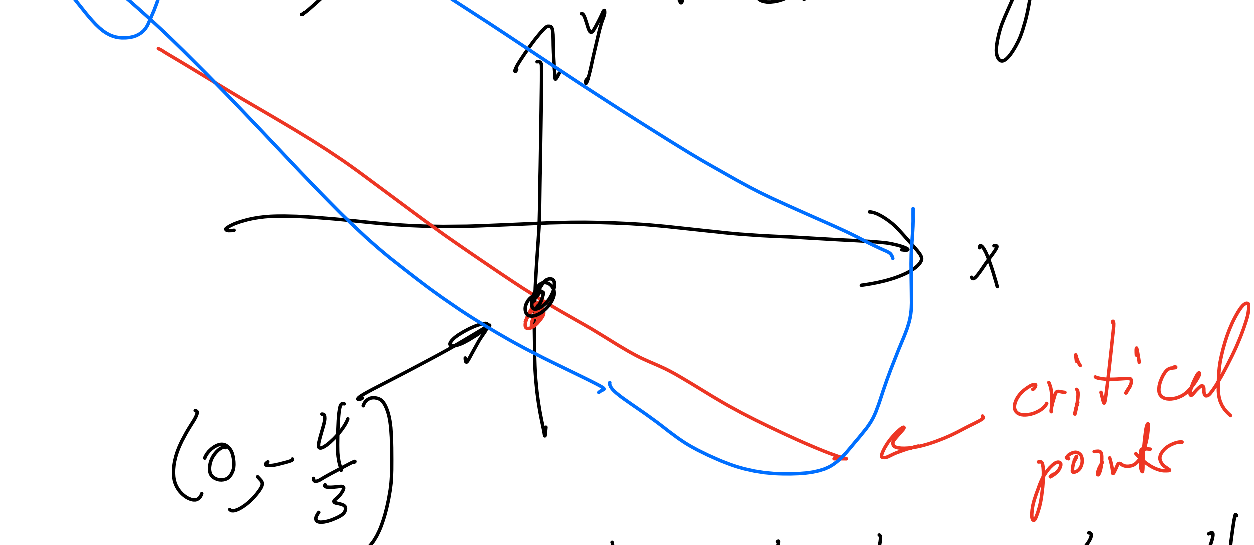
$$f_x = \underset{\substack{\uparrow \\ \text{power rule}}}{2}(2x+3y+4)\underset{\substack{\uparrow \\ \text{derivative of inside}}}{2}$$

$$f_y = 2(2x+3y+4)3$$

$f_x = f_y = 0 \iff 2x + 3y + 4 = 0$

$y = -\frac{2}{3}x - \frac{4}{3}$

$\implies$ lots of critical points



Do 2nd derivative test at $\left(0, -\frac{4}{3}\right)$

$$H = \begin{bmatrix} 8 & 12 \\ 12 & 18 \end{bmatrix}$$

$$\det H = 8(18) - (12)(12)$$
$$= 144 - 144 = 0$$

Examples Where Second Derivative Test Fails

- Consider the function

$$f(x, y) = xy(x^2 - y^2) = xy(x + y)(x - y)$$

  - Its gradient is

$$\begin{aligned}\vec{\nabla} f &= \langle y(x^2 - y^2) + 2x^2y, x(x^2 - y^2) - 2xy^2\rangle \\ &= \langle y(3x^2 - y^2), x(x^2 - 3y^2)\rangle \\ &= \langle ((y(\sqrt{3}x - y)(\sqrt{3}x + y), x(x - \sqrt{3}y)(x + \sqrt{3}y))\rangle\end{aligned}$$

  - Only critical point is $(0, 0)$
  - $H = \begin{bmatrix} 6xy & 3x^2 - 3y^2 \\ 3x^2 - 3y^2 & -6xy \end{bmatrix}$
  - $\det H(0, 0) = 0$
  - The contour $f = 0$ is given by the equation

$$xy(x + y)(x - y)$$

    and consists of the lines $x = 0$, $y = 0$, $y = x$, and $y = -x$

Complicated Example of Second Derivative Test

- $f(x,y) = x^4 + y^4 - 4xy + 1$
  - $\vec{\nabla} f = \langle 4x^3 - 4y, 4y^3 - 4x \rangle$
  - Solve for critical points: $x^3 = y$ and $y^3 = x$
  - Substitute first inequation into second: $x^9 = x$
  - Factor

$$\begin{aligned} 0 &= x^9 - x = x(x^8 - 1) \\ &= x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1) \\ &= x(x - 1)(x + 1)(x^2 + 1)(x^4 + 1) \end{aligned}$$

  - There are three possible values for x: $-1, 0, 1$
  - Since $y = x^3$, the critical points are $(-1,-1)$, $(0,0)$, $(1,1)$

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

$$f_x = 4x^3 - 4y \qquad f_{xx} = 12x^2$$

$$f_y = 4y^3 - 4x \qquad f_{xy} = -4$$

$$f_{yy} = 12y^2$$

Critical points

$$4x^3 - 4y = 0 \text{ and } 4y^3 - 4x = 0$$

$$\Longleftrightarrow \quad y = x^3 \text{ and } x = y^3$$

$$\Longrightarrow \quad x = y^3 = (x^3)^3 = x^9$$

$$\Longrightarrow \quad x^9 - x = 0$$

$$x(x^8 - 1) = 0$$

$$x(x^4 - 1)(x^4 + 1) = 0$$

$$x\left(x^2-1\right)\underbrace{\left(x^2+1\right)}_{\neq 0}\underbrace{\left(x^4+1\right)}_{\neq 0}=0$$

$$x(x-1)(x+1)\left(x^2+1\right)\left(x^4+1\right)=0$$

$\Rightarrow$ only possible critical points are

$x=0$ and $y=0$

or

$x=1$ and $y=1$

or

$x=-1$ and $y=-1$

$\Rightarrow$ $(0,0)$, $(1,1)$, $(-1,-1)$

are only critical points

$$H=\begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$$

At $(0,0)$

$$H(0,0) = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}$$

$$\det H = 0(0) - (-4)^2$$
$$= -16 < 0$$

$\Rightarrow (0,0)$ is a saddle point

At $(1,1)$

$$H = \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix}$$

$$\det H = 12(12) - (-4)^2$$
$$= 144 - 16 > 0$$

$f_{xx} = 12 > 0 \Rightarrow (1,1)$ local min

At $(-1,-1)$

$$H = \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix}$$

$\Rightarrow (-1,-1)$ is local min

Local min, local max,
saddle point
are used to study
shape of the graph of f

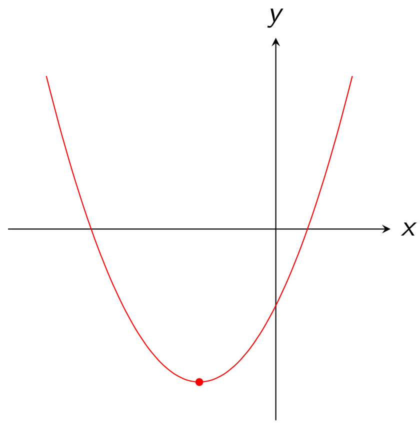
Complicated Example of Second Derivative Test

- $f(x,y) = x^4 + y^4 - 4xy + 1$
- $\vec{\nabla} f = 4\langle x^3 - y, y^3 - x\rangle$
- Critical points are $(-1,-1),(0,0),(1,1)$
- $H = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$
- At the critical points $(-1,1)$ and $(1,1)$
  - $\det H(-1-1) = \det H(1,1) = \det \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix} = 144 - 16 > 0$
  - $H_{11}(-1,-1) = H_{11}(1,1) = 12 > 0$
  - The critical points $(-1,-1)$ and $(1,1)$ are local minima
- At the critical point $(0,0)$
  - $\det H(0,0) = \det \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix} = -16$
  - The critical point $(0,0)$ is a saddle point

Global Optimization

- Consider a function $f(x, y)$ on a domain D in 2-space
- A point (x_0, y_0) is a global maximum point, if $f(x, y) \leq f(x_0, y_0)$ for every $(x, y) \in D$. $f(x_0, y_0)$ is the global maximum value.
- There is at most one maximum value but there can be any number, including infinitely many, maximum points
- A point (x_0, y_0) is a global minimum point, if $f(x, y) \leq f(x_0, y_0)$ for every $(x, y) \in D$. $f(x_0, y_0)$ is the global minimum value.
- There is at most one minimum value but there can be any number, including infinitely many, minimum points
- If D has no boundary, then global optimum points are all critical points
- If D has a boundary then global optimum points are either critical points or boundary points

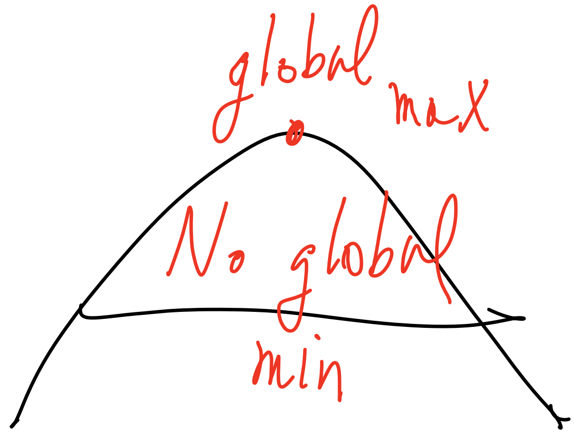
Global Optimization on the Real Line

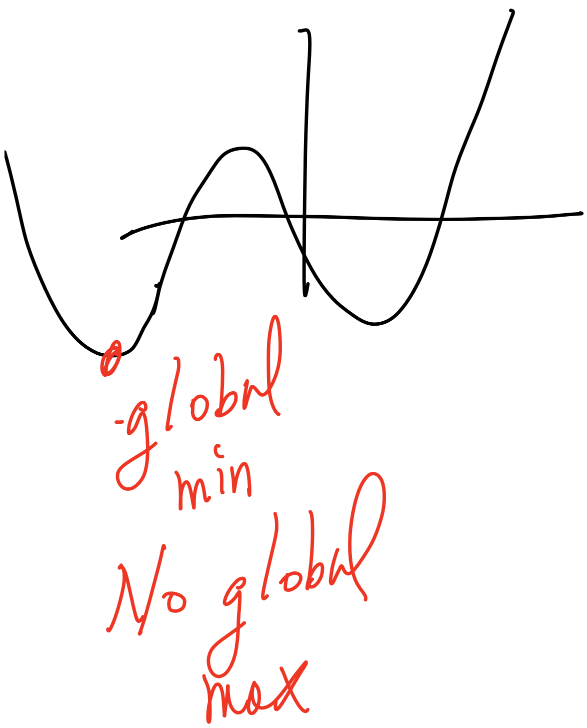


- Suppose $f(x)$ is a smooth function on the entire real line
- Optimal values, if they exist, must occur at a critical point
- To find optima:
  - Study what happens when $x \to \pm\infty$
  - Find all critical points and calculate f at each of them
- In picture:
  - $f(x) \to +\infty$ as $x \to \pm\infty$, which implies that f has no maximum value
  - f is bounded from below, which means that it has a minimum value
  - There is only one critical point, so that has to be the minimum

Global optimization of f(x)

1) Domain of f is whole real line





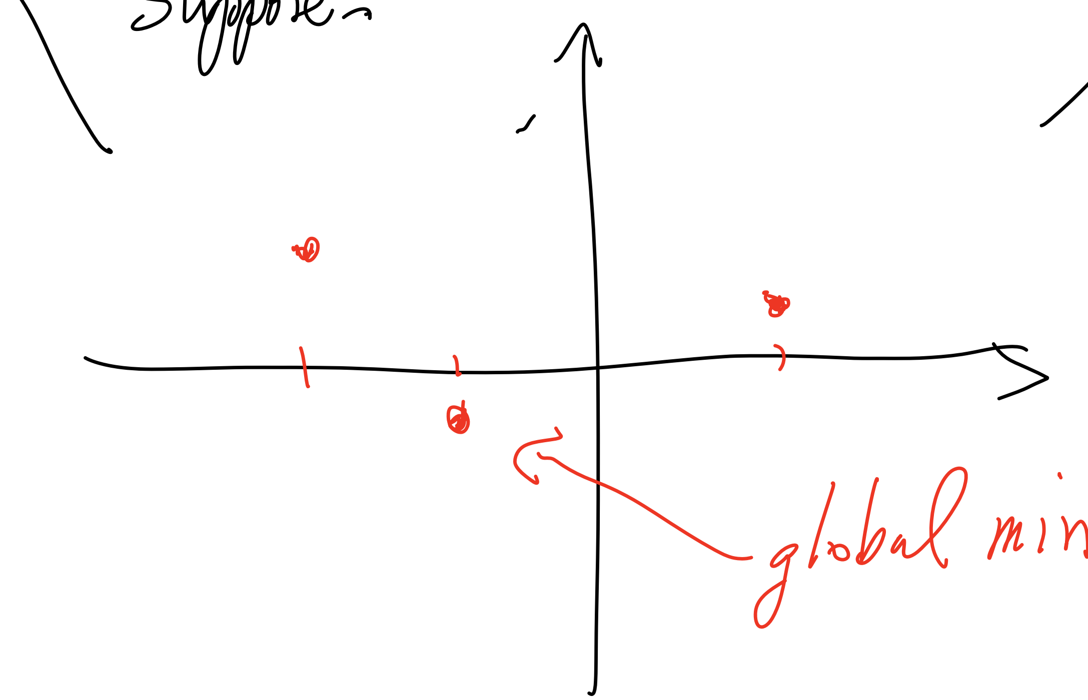
Strategy

1) Look at what happens to f as $x \to +\infty$ and $x \to -\infty$

2) Look for critical points

3) Calculate values of f at critical points

Suppose:



Global Optimization in 2-Space

- Find rectangular cardboard box without a top that encloses a given volume V but using the minimum amount of cardboard
- If dimensions of box are H by W by D, then

$$\text{Volume } V = HWD$$

$$\text{Area of card board } A = 2(HW + HD) + WD$$

- V is constant, and we want to minimize A
- Elminiate one variable $H = \frac{V}{WD}$:

$$A(W, D) = 2\frac{V}{WD}(W + D) + WD = 2V(\frac{1}{D} + \frac{1}{W}) + WD$$

Optimal Cardboard Box

- $A(W, D) = 2V(\frac{1}{D} + \frac{1}{W}) + WD$
- Solution must be at a critical point of A
- Find critical points:

$$A_W = -\frac{2V}{W^2} + D = 0, \qquad A_D = -\frac{2V}{D^2} + W = 0$$

$$D = \frac{2V}{W^2}, \qquad W = \frac{2V}{D^2} = 2V\frac{W^4}{4V^2} = \frac{W^4}{2V}$$

- Therefore,

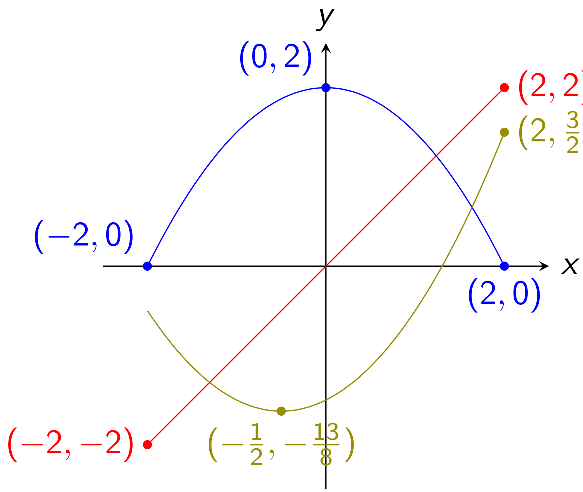
$$0 = \frac{W^4}{2V} - W = W\left(\frac{W^3}{2V} - 1\right)$$

- Since $W \neq 0$,

$$W = (2V)^{1/3} \qquad D = \frac{2V}{W^2} = (2V)^{1/3}$$

$$H = \frac{V}{WD} = \frac{V}{(2V)^{2/3}} = 2(2V)^{1/3}$$

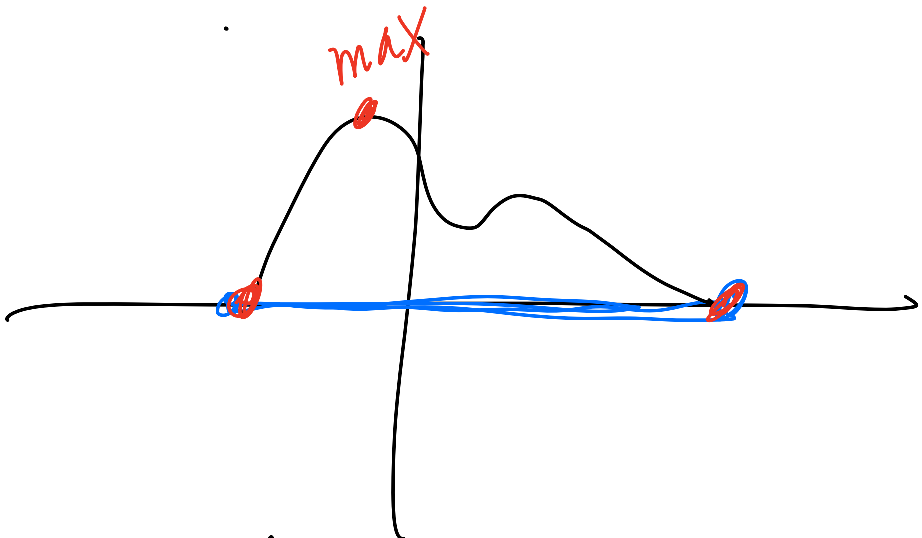
Global Optimization on a Bounded Interval



- The global optima of a smooth function on a bounded closed interval are always at critical or end points
- Here, we have three functions:

$$
\begin{aligned}
f(x) &= 2 - \frac{1}{2}x^2 \\
g(x) &= x \\
h(x) &= \frac{1}{2}(x^2 - x - 3)
\end{aligned}
$$

2) Domain of f is closed bounded interval

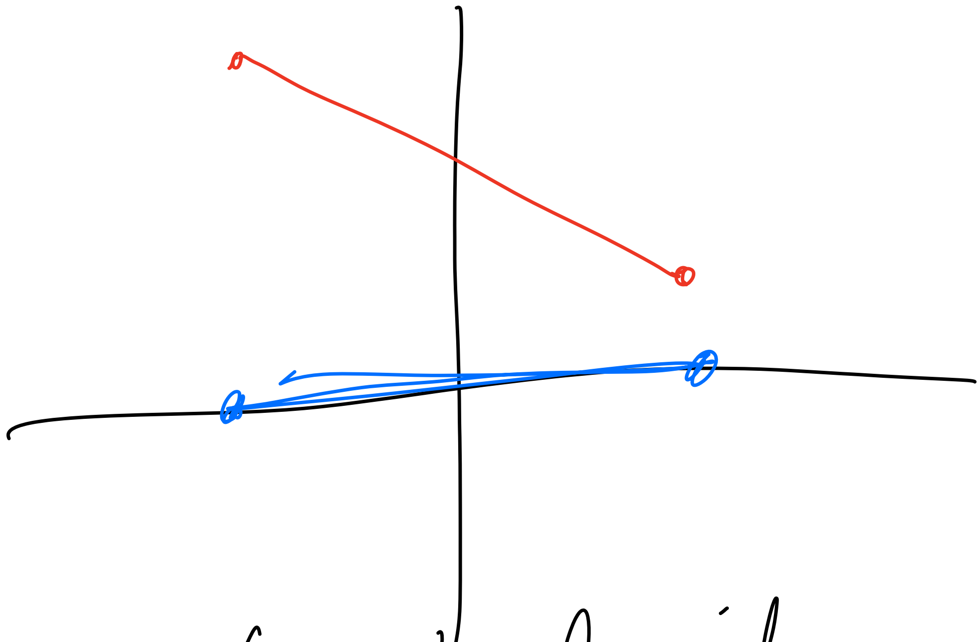


Strategy

1) Find all critical points
2) Calculate f at critical points and boundary points
3) The point with least value of f is the global minimum

etc.

Interesting case: f linear



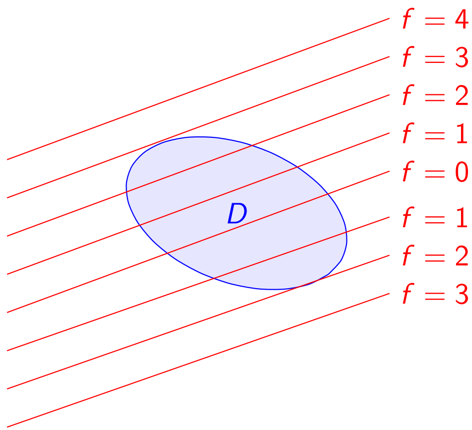
No critical points

Max, min at boundary

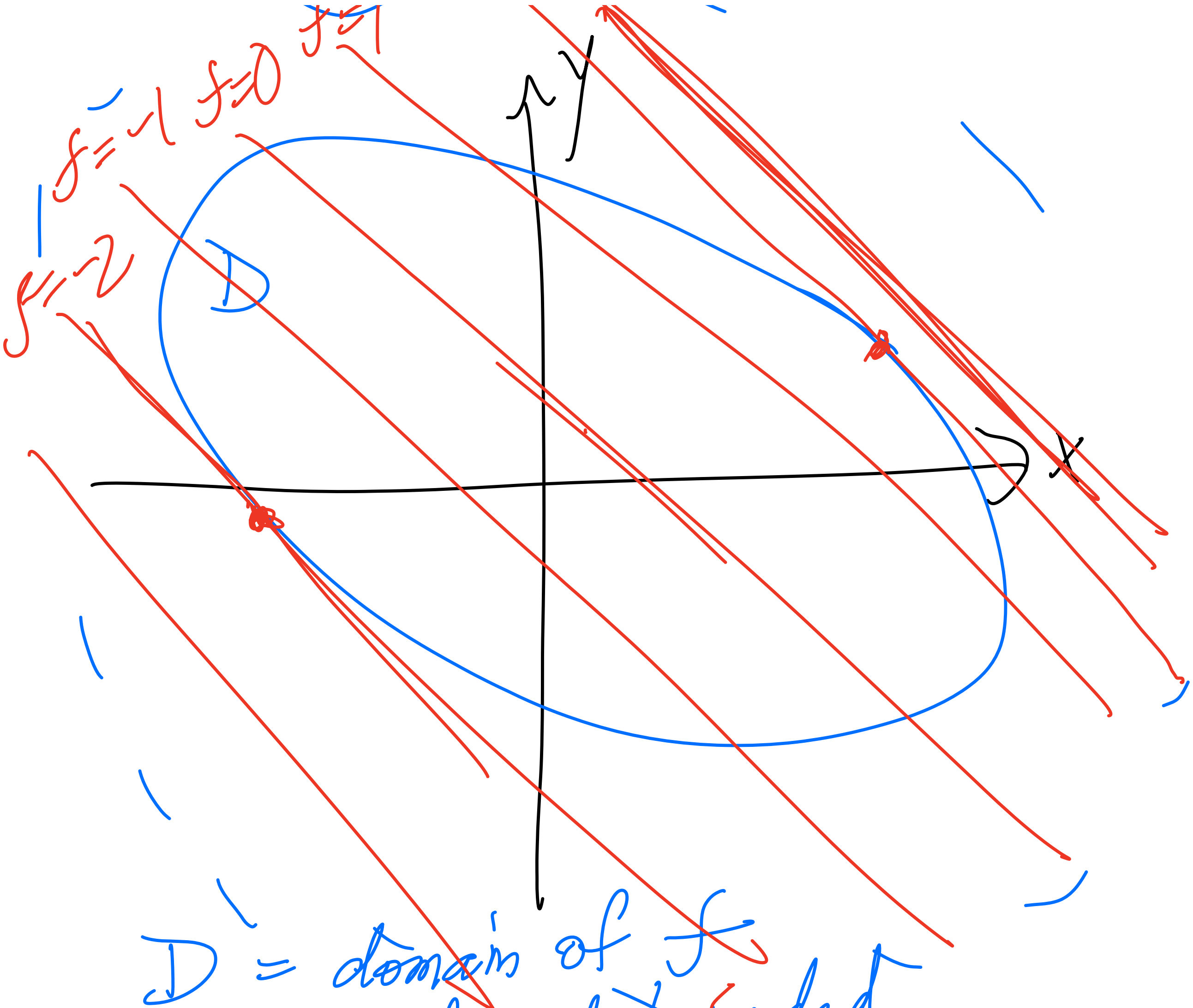
Finding Optimal Values and Points on an Interval

- Find all of the critical points that lie in the interval
- Calculate the value of the function at each critical and each end point
- Identify where the function is maximum and where it is minimum

Global Optima on a Bounded Domain in 2-Space



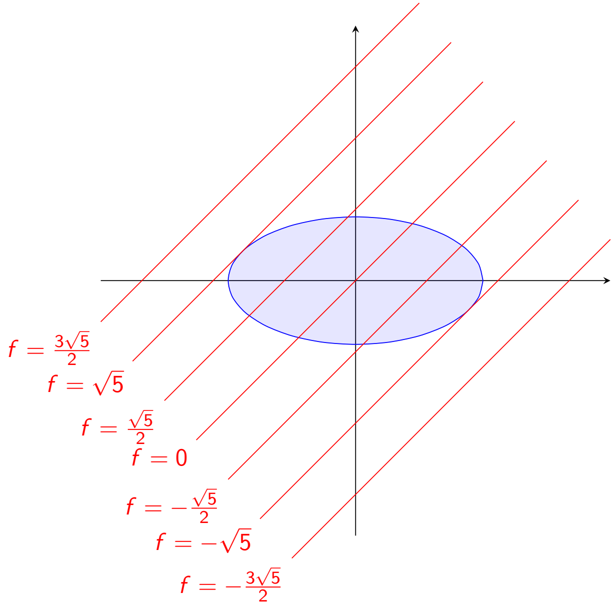
- Suppose $D = \{(x, y) \ : \ g(x, y) \leq 1\}$
- Maximize or minimize $f(x, y)$ with (x, y) restricted to the domain D
- An optimal point must be either a critical point or a point on the boundary
- If optimal point is on bondary, then it must be at a point where the contour of f and the boundary are tangent
  - Where $\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0)$ for some scalar λ



D = ~~domain~~ of ~~f~~ f
is closed and ~~bounded~~,

Suppose f is linear
Contours for evenly spaced values
of f are parallel straight
lines

Example



- Optimize $f(x, y) = y - x$ over all (x, y) such that $\frac{x^2}{4} + y^2 \leq 1$
- Since $\vec{\nabla} f = \langle -1, 1 \rangle$, there are no critical points
- The boundary is the contour $g = 1$, where $g(x, y) = \frac{x^2}{4} + y^2$
- Solve for x, y, λ such that

$$\vec{\nabla} f(x, y) = \lambda \vec{\nabla} g(x, y) \text{ and } g(x, y) = 1$$

Constrained Optimization Example

- Constraint: $g = 1$, where $g(x, y) = \frac{x^2}{4} + y^2$
- Objective function: $f(x, y) = y - x$
- Solve for (x, y) and λ such that $\vec{\nabla} f = \lambda \vec{\nabla} g$

$$\langle -1, 1 \rangle = \lambda \langle \frac{x}{2}, 2y \rangle$$

- $\lambda \neq 0$ because left side is nonzero
- Therefore,

$$\begin{aligned}
\langle -\lambda^{-1}, \lambda^{-1} \rangle &= \langle \frac{x}{2}, 2y \rangle \\
2y &= -\frac{x}{2} \\
y &= -\frac{x}{4} \\
1 &= \frac{x^2}{4} + \frac{x^2}{16} = \frac{5}{16} x^2 \\
x &= \pm \frac{4}{\sqrt{5}}
\end{aligned}$$

Constrained Optimization Example

- Constraint: $g = 1$, where $g(x, y) = \frac{x^2}{4} + y^2$
- Objective function: $f(x, y) = y - x$
- Solve for (x, y) and λ such that $\vec{\nabla} f = \lambda \vec{\nabla} g$
- $y = -\frac{x}{4}$ and $x = \pm\frac{4}{\sqrt{5}}$
- Therefore,

$$(x, y) = (\frac{4}{\sqrt{5}}, -\frac{1}{\sqrt{5}}) \text{ or } (-\frac{4}{\sqrt{5}}, \frac{1}{\sqrt{5}})$$

- Calculate values of f

$$f(\frac{4}{\sqrt{5}}, -\frac{1}{\sqrt{5}}) = -\sqrt{5} \text{ and } f(-\frac{4}{\sqrt{5}}, \frac{1}{\sqrt{5}}) = \sqrt{5}$$

- The constrained maximum value of f is $\sqrt{5}$ and occurs at $(x, y) = (-\frac{4}{\sqrt{5}}, \frac{1}{\sqrt{5}})$
- The constrained minimum value of f is $-\sqrt{5}$ and occurs at $(x, y) = (\frac{-4}{\sqrt{5}}, -\frac{1}{\sqrt{5}})$

Optimization on a Bounded Domain

- Suppose you want to find the maximum or minimum value of a function f on a closed bounded domain D in 2-space
- Closed means D contains its boundary
- The maximum and minimum points of f must either be critical points in D or lie on the boundary of D
- To find the optimal points and corresponding values of f:
  - Find all critical points of f that lie in D
  - Find all maximum or minimum points on the boundary D by doing constrained optimization
  - Calculate the value of f on each point identified in previous steps

Constrained Optimization on a Contour

- Objective function $f(x, y)$
- Constraint equation $g(x, y) = c$, where c is a constant
- Assume
  - The contour $g = c$ is bounded
  - $\vec{\nabla} g(x, y) \neq 0$ for any (x, y) in the contour $g = c$
- The constrained maxima and minima must occur at points in the contour that are either critical points of f or where $\vec{\nabla} f$ and $\vec{\nabla} g$ point in the same or opposite directions, i.e.

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

- Note that $\lambda = 0$ corresponds to a critical point of f
- Solution process:
  - Find all points (x, y) such that $g(x, y) = 0$ and there is a scalar λ such that $\vec{\nabla} f(x, y) = \lambda \vec{\nabla} g(x, y)$
  - Calculate f at all points found in previous step
  - Identify maximum or minimun points and values