MATH-UA 148 Honors Linear Algebra
Adjoint of Linear Map
Orthogonal Transformations

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LIVE TRANSCRIPT

Adjoint of a Linear Transformation

- If V is an inner product space and $L : V \to V$ is a linear map, then there is a linear map $L^* : V \to V$, called the adjoint of L, that satisfies
$$\langle L(v), w\rangle = \langle v, L^*(w)\rangle \text{ for any } v, w \in V$$
- If $V = \mathbb{R}^n$, $\langle \cdot, \cdot \rangle$ is the standard dot product, and
$$L(v) = Mv,$$
where M is an n-by-n matrix, then
$$\begin{aligned}\langle w, L(v)\rangle &= w \cdot Mv \\ &= \langle e_i w^i, e_j M^j_k v^k\rangle \\ &= \sum_{j=1}^{n} w^j M^j_k v^k = v^k(M^j_k w^j) \\ &= v \cdot M^t w\end{aligned}$$
- Therefore,
$$L^*(w) = M^t w$$

Properties of Adjoint Map

- $(a^1 L_1 + a^2 L_2)^* = a^1 L_1^* + a^2 L_2^*$
- $(L_1 \circ L_2)^* = L_2^* \circ L_1^*$, because for any $v, w \in V$,

$$\begin{aligned}\langle (L_1 \circ L_2)(v), w \rangle &= \langle L_1(L_2(v)), w \rangle \\ &= \langle L_2(v), L_1^*(w) \rangle \\ &= \langle v, L_2^*(L_1^*(w)) \rangle \\ &- \langle v, (L_2^* \circ L_1^*)(w) \rangle\end{aligned}$$

Adjoint Map With Respect to Basis (Part 1)

- Let $(e_1, \ldots, e_n)$ be a basis of V and let

$$A_{jk} = \langle e_j, e_k \rangle$$

- Let $L : V \to V$ be a linear map such that

$$L(e_i) = e_k M_i^k$$

- If $v = e_j a^j$ and $w = e_k b^k$, then

$$\begin{aligned}
\langle L(v), w \rangle &= \langle L(a^j e_j), e_k b^k \rangle = \langle a^j L(e_j), e_k b^k \rangle \\
&= \langle a^j M_j^i e_i, e_k b^k \rangle = (M_j^i a^j) \langle e_i, e_k \rangle b^k \\
&= (M_j^i a^j) A_{ik} b^k = (Ma) \cdot Ab \\
&= a \cdot M^t A b \ = a \cdot A(A^{-1} M^t A b) \\
&= a \cdot A(M^* b) = \langle v, L^*(w) \rangle,
\end{aligned}$$

where $M^* = A^{-1} M^t A$ and $L^*(e_j) = e_k (M^*)_j^k$

Adjoint Map With Respect to Basis

- The adjoint map is therefore given by

$$L^*(e_k a^k) = e_j (M^*)^j_k a^k,$$

where

$$(M^*)^j_k = (A^{-1})^{jp} M^q_p A_{qk}$$

- If $(e_1, \ldots, e_n)$ is an orthonormal basis, then $A = I$ and

$$(M^*)^j_k = M^k_j = (M^t)^k_j$$

Orthogonal Transformation

- A map $F : V \to V$ is **orthogonal** if for any $v, w \in V$,

$$\langle F(v), F(w) \rangle = \langle v, w \rangle$$

- In other words, F preserves the inner product
- Consequences:
  - F is linear
  - F is bijective, because if F is linear and $F(v) = 0$, then

$$0 = \langle F(v), F(v) \rangle = \langle v, v \rangle$$

- Therefore,

$$\langle v, w \rangle = \langle F(v), F(w) \rangle = \langle v, F^*(F(w)) \rangle$$

- It follows that F is orthogonal if and only if

$$F^* \circ F = I$$

An Orthogonal Map is Linear

- Let $(e_1, \ldots, e_n)$ be an orthonormal basis
- For each $1 \leq k \leq n$, let $f_k = F(e_k)$
- For any $1 \leq j, k \leq n$,

$$\langle f_j, f_k \rangle = \langle F(e_j), F(e_k) \rangle = \langle e_j, e_k \rangle = \delta_{jk}$$

- Therefore $(f_1, \ldots, f_n)$ is also an orthonormal basis
- For any $v = a^k e_k$ and $1 \leq j \leq n$,

$$\begin{aligned}\langle F(v), f_j \rangle &= \langle F(a^k e_k), F(e_j) \rangle \\ &= \langle a^k e_k, e_j \rangle \\ &= a^k \langle e_k, e_j \rangle \\ &= a^j,\end{aligned}$$

  which implies that for all $1 \leq k \leq n$,

$$F(a^k e_k) = a^k f_k = a^k F(e_k)$$

- This implies that F is linear

Orthogonal Matrix

- If $V = \mathbb{R}^n$ and the inner product is the dot product, then a linear map $L(v) = Mv$ is orthogonal if and only if the matrix M satisfies
$$M^t M = I$$
- Recall that if $C_1, \ldots, C_n$ are the columns of M, then they are the rows of M^t and
$$(M^t M)_{jk} = C_j \cdot C_k$$
- Therefore, a matrix is orthogonal if and only if its columns are an orthonormal basis of $\mathbb{R}^n$
- If M is orthogonal, then $M^{-1} = M^t$ and therefore
$$MM^t = MM^{-1} = I$$
- It follows that M is orthogonal if and only if its rows are an orthonormal basis of $\mathbb{R}^n$

2-Dimensional Orthogonal Matrices

- A matrix
$$M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$
is orthogonal if
$$a^2 + b^2 = c^2 + d^2 = 1 \text{ and } ad + bc = 0$$
- This holds if and only if
$$(c, d) = \pm(-b, a)$$
- Therefore, an orthogonal matrix is of the form
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ or } \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

Orthogonal Group

- Let $O(n)$ be the set of all n-by-n orthogonal matrices
- It has the following properties:
  - If $A, B \in O(n)$, then $AB \in O(n)$
  - $I \in O(n)$
  - If $A \in O(n)$, then $A^{-1} = O(n)$
- More generally, a subset $G \subset \mathcal{M}_{n \times n}$ is called a **matrix group** if the following properties hold:
  - If $A, B \in G$, then $AB \in G$
  - $I \in G$
  - If $A \in G$, then $A^{-1} = G$
- We therefore call $O(n)$ the **orthogonal group**

Special Orthogonal Group

- Observe that if $A \in O(n)$, then

$$1 = \det I = \det A^t A = (\det A^t)(\det A) = (\det A)^2$$

- Therefore, $\det A = \pm 1$
- The **special orthogonal group** is defined to be

$$SO(n) = \{A \in O(n) \ : \ \det A = 1\}$$

Transformation Group

- Given a vector space V, let $\mathrm{GL}(V)$ be the set of all invertible linear transformations
- A subset $G \subset \mathrm{GL}(v)$ is a **transformation group** if it satisfies the following properties
  - $L_1, L_2 \in G \implies L_1 \circ L_2 \in G$
  - $I \in G$
  - $L \in G \implies L^{-1} \in G$
- $GL(V)$ itself is a transformation group

Group of Orthogonal Transformations

- If V is an inner product space, then the set $O(v)$ of all orthogonal transformations is a transformation group