MATH-UA 325 Analysis I

Basic Set Theory
Deductive Logic

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Definition of Mathematics

- Deriving new knowledge from old using deductive logic and abstraction
- You have to start somewhere
  - Axioms
  - Definitions
  - Assumptions
- New knowledge
  - Theorems
  - Propositions
  - Lemmas
- Steps used to derive new knowledge from old
  - Precise logical deductions

Grading

- Homework: 30%
- Midterm: 30%
- Final: 40%
- Tweaks

Textbook

- Lebl, Basic Analysis: Introduction to Real Analysis
    - Can be downloaded legally for free
- Other sources that you can consult and are available for free
    - Abbott, Understanding Analysis, Springer
    - Katz, Calculus for Cranks
    - Pugh, Real Mathematical Analysis, Springer
    - Tao, Analysis I
    - Anything else you can find on the internet

Advice

- This is a hard course
    - This is my first time teaching this course
    - Reserve a lot of time and energy for this course
- Focus on homework
    - Problems are hard
    - Use precise definitions and rigorous logic
        - Theorems are sometimes useful
    - Lectures are relatively useless because proofs in homework are easier than proofs in lectures
    - Look for useful examples and counterexamples
    - Try to prove easy cases and identify where the hard step is
- Exams are even easier
    - Due to time limit

0. Preliminaries

0.3 Basic Set Theory

- A **set** is a collection of zero or more things
- Examples
  - The set of all dogs living in New York City
  - The set whose members are $-17, 15, 234$
  - The set with nothing in it
- Notation
  - A thing is usually represented by a Roman or Greek letter
  - A set is usually represented by a Roman or Greek letter
  - A set can be described as follows:
    - English sentence

$$\Sigma \text{ is the set of all dogs living in NYC}$$

    - Mathematical sentence

$$\Sigma = \{\text{Complete list or description of things}\}$$

Set Membership

- $\alpha \in W$ means the following:
    - W is a set
    - α is a thing
    - α is a member of the set W
- $W \in \alpha$ means
    - α is a set
    - W is a thing
    - W is a member of the set α
- $W \notin \alpha$ means
    - α is a set
    - W is a thing
    - W is not a member of the set α

Numbers

- Numbers
  - Natural numbers: $\mathbb{N} = \{1, 2, 3, \dots\}$
    - Operations: Addition, multiplication
    - Properties: Ordering, Associativity, Commutativity, Distributivity
  - Integers: $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
    - Additional property: Each integer has an additive inverse
  - Rationals: $\mathbb{Q}$ is the set of fractions
    - Additional property: Each nonzero rational has a multipicative inverse
  - Reals: $\mathbb{R}$
    - Discussed in detail later

Language of Sets

- A set with one element
  - x is not the same as $\{x\}$
- $A \subset B$
  - A is a subset of B
  - Every element of A is also an element of B
  - Example
    - $\mathbb{N} \subset \mathbb{Q}$
- $A \cup B$
  - The union of A and B
  - The set whose members consist of the members of A and the members of B
  - Includes, but only once, elements that are in both A and B
- $A \backslash B$
  - The **difference** of A and B (order matters!)
  - The set of elements in A that are not in B

Union and Intersection of Infinite Collection of Sets

- Suppose $A_1, A_2, \dots$ is an infinite collection of setes
- The union of all sets in the collection is written as and defined by

$$\bigcup_{n=1}^{\infty} A_n = \{x \ : \ \exists n \in \mathbb{N} \text{ such that} x \in A_n\}.$$

  It's possible for x to be in more than one of the sets
- The intersection of all sets in the collection is written as and defined by

$$\bigcap_{n=1}^{\infty} A_n = \{x \ : \ \forall n \in \mathbb{N}, \ x \in A_n\}.$$

Union and Intersection of Doubly Indexed Collection of Sets

- Let $A_{j,k},\ 1 \le j, k < \infty$ be a collection of sets
- Union of intersections

$$\bigcup_{j=1}^{\infty} \bigcap_{k=1}^{\infty} A_{j,k} = \{x \ : \ \exists j \in \mathbb{N} \text{ such that } x \in \bigcap_{k=1}^{\infty} A_{j,k}\}$$

$$= \{x \ : \ \exists j \in \mathbb{N} \text{ such that } \forall k \in \mathbb{N}, x \in A_{j,k}\}$$

- Intersection of unions

$$\bigcap_{j=1}^{\infty} \bigcup_{k=1}^{\infty} A_{j,k} = \{x \ : \ \forall j \in \mathbb{N},\ x \in \bigcup_{k=1}^{\infty} A_{j,k}\}$$

$$= \{x \ : \ \forall j \in \mathbb{N},\ \exists k \in \mathbb{N} \text{ such that } x \in A_{j,k}\}$$

- Example: Let $A_{j,k} = [0, j] \times [0, k]$

Ordered Pairs and Cartesian Product

- If A and B are sets, then an **ordered pair** is a pair that satisfies the following:
  - The first element is an element of A and the second element is an element of B
  - $(a, b) \neq (b, a)$ unless $a = b$
- The ordered pair of $a \in A$ and $b \in B$ is denoted (a, b)
- The **Cartesian product** of A and B is the set of all possible ordered pairs

$$A \times B = \{(a, b) \text{ for every } a \in A \text{ and } b \in B\}$$

Deductive Logic

English Sentences

- A English sentence always contains

Subject verb (possibly an object

- Examples of English sentences
  - My name is Smith
  - All cats are black
  - He drove his car to the store
- The following are not English sentences
  - The stores that sell toys
  - Coming from the store

Mathematical Sentences

- A mathematical sentence has the same structure except that they are assertions about mathematical things
- Forms of a mathematical sentence
  - An English sentence without any symbols
    - A set with one element is not the same as the element itself
  - An English sentence with symbols
    - x is not equal to $\{x\}$
  - Symbols only

$$x \neq \{x\}$$

- Always write in complete mathematical sentences
- Fragments of sentences (i.e., phrases) often lead to errors in calculations and proofs

True or False

- In principle, every mathematical sentence is either true or false
    - Impossible to be both
    - Impossible to be neither
- In practice, the following are possible
    - You know the sentence is true
    - You know the sentence is false
    - You do not know whether the sentence is true or false
    - The sentence is ambiguous

And, Or

- Let A and B be mathematical sentences
- A and B is the sentence that
    - Is true if both A and B are true
    - Is false otherwise
- A or B is the sentence that
    - Is false only if both A and B are false
    - Is true otherwise
    - has a meaning different from its everyday English meaning
        - "Tom or Jerry did it" means one of the two, but not both, did it
        - "$x < 1$ or $x > -1$" means one or both of the inequalities is true

Negation

- not A is the sentence that
  - Is true if A is false
  - Is false if A is true
- not(not A) is equivalent to A
- not(A or B) is equivalent to (not A) and(not B)
  - Therefore,

    $$\text{not}((\text{not } A) \text{ and}(\text{not } B))$$

    is equivalent to A or B
- The sentence or(A and B) is equivalent to (not A) or(not B)
  - Therefore,

    $$\text{not}((\text{not } A) \text{ or}(\text{not } B))$$

    is equivalent to A and B

Implication

- An **implication** is a conditional statement

$$\text{If } S\text{, then } T,$$

  where S and T are mathematical sentences
- Equivalent forms of this sentence
  - T if S
  - T is a consequence of S
  - $S \implies T$
- S states the assumptions
- T states the conclusion or consequence
- Its truth or falsity is determined as follows:

S	T	$S \implies T$
T	T	T
F	T	T
T	F	F
F	F	T

Modus Ponens

- Let A, B, and C be mathematical sentences
- Suppose you know only the following:
  - $A \implies B$
  - A
- Then B is true
- Suppose you know only the following:
  - $A \implies B$
  - $B \implies C$
- Then $A \implies C$ is true
- Every step of a proof is one of these
- This is known as modus ponens

Other possibilities

- Suppose you know only the following:
  - $A \implies B$
  - not A
- Then no conclusion about B can be inferred
- Suppose you know only the following:
  - $A \implies B$
  - B
- Then no conclusion about A can be inferred
- Suppose you know only the following:
  - $A \implies B$
  - not B
- Then A is false
- Suppose you know only the following:
  - $A \implies B$
  - Nothing about A or B
- Then no conclusion about A or B can be inferred

Equivalence, Contrapositive

- A and B are **equivalent** if

$$(A \implies B) \text{and} (B \implies A)$$

  - This is also expressed as follows:
    - A if and only if B
    - A iff B
    - $A \iff B$
- The *contrapositive* of

$$A \implies B$$

is

$$\text{not}\, B \implies \text{not}\, A$$

- The two sentences are equivalent
- In other words, the following sentence is always true

$$(A \implies B) \iff (\text{not}\, A \implies \text{not}\, B)$$

Converse

- The converse of $A \implies B$ is $B \implies A$
- $A \implies B$ is NOT equivalent to $B \implies A$

Quantifiers

- $\forall$ means "for each"
- $\exists$ means "there exists"
- Examples
  - For each integer n, n^2 is odd

$$\forall n \in \mathbb{Z},\ n^2 \text{ is odd}$$

    - Negation: There exists an integer n such that n^2 is not odd

$$\exists n \in \mathbb{Z},\ n^2 \text{ is not odd}$$

  - There exists an integer that is between 0 and 1

$$\exists n \in \mathbb{Z} \text{ such that } 0 < n < 1$$

    - Negation: For each integer n, n does not lie between 0 and 1

$$\forall n \in \mathbb{Z},\ n \leq 0 \text{ or } n \geq 1$$

Nested Quantifiers

- There exists an integer m such that for each integer n less than m, $n^2 > m^2$

$$\exists m \in \mathbb{Z} \text{ such that } \forall n < m,\ n^2 > m^2$$

- For every integer m, there exists an integer n less than m such that $n^2 < m^2$

$$\forall m \in \mathbb{Z}, \exists n < m \text{ such that } n^2 < m^2$$

- For every $\epsilon > 0$, there exists $\delta > 0$ such that

$$1 < x < 1 + \delta \implies 1 < x^2 < 1 + \epsilon$$