MATH-UA 325 Analysis I

Quantifiers
Mathematical Induction
Functions and Maps
Equivalence Relations
Graph of Function

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Quantifiers

- $\forall$ means "for each"
- $\exists$ means "there exists"
- Examples
  - For each integer n, n^2 is odd

$$\forall n \in \mathbb{Z},\ n^2 \text{ is odd}$$

    - Negation: There exists an integer n such that n^2 is not odd

$$\exists n \in \mathbb{Z},\ n^2 \text{ is not odd}$$

  - There exists an integer that is between 0 and 1

$$\exists n \in \mathbb{Z} \text{ such that } 0 < n < 1$$

    - Negation: For each integer n, n does not lie between 0 and 1

$$\forall n \in \mathbb{Z},\ n \leq 0 \text{ or } n \geq 1$$

Nested Quantifiers

- There exists an integer m such that for each integer n less than m, $n^2 > m^2$

$$\exists m \in \mathbb{Z} \text{ such that } \forall n < m,\ n^2 > m^2$$

- For every integer m, there exists an integer n less than m such that $n^2 < m^2$

$$\forall m \in \mathbb{Z}, \exists n < m \text{ such that } n^2 < m^2$$

- For every $\epsilon > 0$, there exists $\delta > 0$ such that

$$1 < x < 1 + \delta \implies 1 < x^2 < 1 + \epsilon$$

0.3.2 Mathematical Induction

- Suppose there is an infinite sequence of sentences S_n, $n \in \mathbb{N}$, that you want to prove
- A proof by mathematical induction proceeds as follows:
  - First, prove S_1 directly
  - Assume you have proved $S_1, \ldots, S_n$
  - Prove S_{n+1}.

Mathematical Induction: Example

- Example: Prove that for any integer $n \geq 1$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \qquad (S_n)$$

- If $n = 1$, the left side is equal to 1, and the right is equal to

$$\frac{1(2)(3)}{6} = 1$$

and therefore (S_1) is true

Mathematical Induction: Example

- Suppose we know that (S_n) is true for $n = 1, \ldots, m$
- It follows that

$$
\begin{aligned}
1^2 + \cdots + m^2 + (m+1)^2 &= \frac{m(m+1)(2m+1)}{6} + (m+1)^2 \\
&= \frac{m(m+1)(2m+1) + 6(m+1)^2}{6} \\
&= \frac{(m+1)(2m^2 + m + 6m + 6)}{6} \\
&= \frac{(m+1)(2m^2 + 7m + 6)}{6} \\
&= \frac{(m+1)(m+2)(2m+3)}{6}
\end{aligned}
$$

- Therefore, S_{m+1} is true
- This proves that S_n is true for all $n \in \mathbb{N}$

0.3.3 Functions and Maps

- Let A and B be sets
- A **function** or **map** P with domain A and range (or codomain) B assigns to ech element of A an element of B, which is usually written

$$P : A \to B$$

- P accepts an input from A and produces an output from B
- Notation: if $a \in A$ is an input, then $P(a) \in B$ is the corresponding output
- In general,

$$(\textit{function name})(\textit{input})$$

  represents the output of *function name* if the input is *input*
- In this course, the domain will almost always be a subset of $\mathbb{R}$ and the codomain will be $\mathbb{R}$

Image and Inverse Image of a Map

- The **image** under a map $M : P \to Q$ of a subset $C \subset P$ is

$$M(C) = \{q \in Q \ : \ \exists p \in C \text{ such that } M(p) = q\}$$

- It is the set of all outputs of M for all inputs in C
- The **inverse image** under M of a set $S \subset Q$ is

$$M^{-1}(S) = \{p \in P \ : \ M(p) \in S\}$$

- It is the set of all inputs whose corresponding outputs are in S
- Example: Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$\forall x \in \mathbb{R},\ f(x) :- \sin(\pi x)$$

  - $f(\mathbb{R}) = [-1, 1]$
  - $f(\{\dots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \dots\}) = \{0\}$
  - $f^{-1}(\{0\}) = \{\dots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \dots\}$

Injective and Surjective Functions and Maps

- A function $F : A \to B$ is **surjective** if

$$F(A) = B$$

  - In other words, for any $b \in B$, there exists $a \in A$ such that $f(a) = b$
- A function $F : A \to B$ is **injective** if for any $b \in F(A)$, the set $F^{-1}(b)$ has exactly one element
  - In other words, if $F(a_1) = F(a_2)$, then $a_1 = a_2$

Composition of Functions

- If $f : A \to B$ and $g : B \to C$, the **composition** of g and f is the function
$$g \circ f : A \to C,$$
where for each $a \in A$,
$$(g \circ f)(a) = g(f(a))$$
- It is important that the image of f is a subset of the domain of g,
$$f(A) \subset \text{domain}(g)$$