MATH-UA 325 Analysis I Fall 2023

Square Root of 2 is Irrational

Reals

Arithmetic and Ordering of Reals

Upper and Lower Bounds

Completeness of $\mathbb{R}$

Square Root of 2 is in $\mathbb{R}$

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The Square Root of 2 is Not Rational

- But there is no rational number r such that $r^2 = 2$
  - Suppose $r \in \mathbb{Q}$ satisfies $r^2 = 2$
  - Since $r \in \mathbb{Q}$, there are integers p, q such that $r = \frac{p}{q}$
  - Assume p and q have no common factors
    - At least one has to be odd
  - Therefore

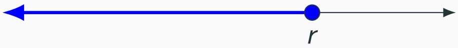
$$\frac{p^2}{q^2} = 2 \implies p^2 = 2q^2$$

    - p^2 is even, so p is even
    - Therefore, p^2 is divisible by 4
    - Therefore, q is even
    - Contradiction

The Set of Rationals Has Holes

- Given $r \in \mathbb{Q}$, let

$$A = \{q \in \mathbb{Q} \ : \ q \leq r\}$$



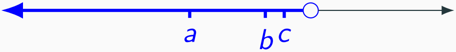
- Let

$$A = \{q \in \mathbb{Q} \ : \ q \leq 0 \text{ or } q^2 \leq 2\}$$

The Reals

- The set of real numbers is denoted $\mathbb{R}$
- They are designed to fill in all of the holes in $\mathbb{Q}$
- The reals can be defined in different ways:
    - Infinite decimals
    - Dedekind cuts

Open Left Half Lines in Q



- A **left half line** in $\mathbb{Q}$ is a nonempty subset $A \subset \mathbb{Q}$ such that the following hold:
  - If $b \in A$, $a \in \mathbb{Q}$, and $a \leq b$, then $a \in A$
- Example: Given $r \in \mathbb{Q}$,

$$(-\infty, r] = \{q \in \mathbb{Q} \ : q \leq r\}$$

- A half line A is **open** if the following holds:
  - If $b \in A$, then $\exists c \in A$ such that $b < c$
- Examples:
  - For each $r \in \mathbb{Q}$, $(-\infty, r)$
  - $\{q \in \mathbb{Q} \ : \ q \leq 0 \text{ or } q^2 \leq 2\}$
- Every rational has an open left half line next to it
- Not every open left half line has a rational next to it
- Idea: Define the real numbers using open left half lines

Closed Right Half Lines

- A **right half line** in $\mathbb{Q}$ is a nonempty subset $B \subset \mathbb{Q}$ such that the following hold:
    - If $b \in B$, $a \in \mathbb{Q}$, and $a \geq b$, then $a \in B$
- Example: Given $r \in \mathbb{Q}$,

$$(-\infty, r] = \{q \in \mathbb{Q} \ : q > r\}$$

- A half line A is **closed** if there exists $m \in \mathbb{Q}$ such that $q \geq m$ for each $q \in \mathbb{Q}$
- Examples:
    - $\{q \in \mathbb{Q} \ : \ q^2 \geq 4\}$ is a closed right half line
    - $\{q \in \mathbb{Q} \ : \ q^2 \geq 2\}$ is not a closed right half line

Dedekind Cuts

- A **cut** is a pair of nonempty disjoint subsets $A, B \subset \mathbb{Q}$, denoted $(A|B)$ such that
  - A is an open left half line
  - $B = \mathbb{Q} \backslash A$
- For each $r \in \mathbb{Q}$, there is a unique cut $(A|B)$ such that

  $$A = \{q \in \mathbb{Q} \ : \ q < r\} \text{ and } B = \{q \in \mathbb{Q} \ : \ q \geq r\},$$

  where B is a closed left half line
- On the other hand, if $(A|B)$ is the cut such that

  $$A = \{q \in \mathbb{Q} \ : \ q \leq 0 \text{ or } q^2 < 2\} \text{ and } B = \{q \in \mathbb{Q} \ : \ q^2 \geq 2\},$$

  then B is not a closed left half line

The Reals

- Define the set of real numbers to be the set of all cuts in $\mathbb{Q}$
- Ordering of reals
  - $(A|B) = (C|D)$ if $A = C$
  - $(A|B) \leq (C|D)$ if $A \subset C$
  - $(A|B) < (C|D)$ if $(A|B) \leq (C|D)$ and $(A|B) \neq (C|D)$
  - The following hold:
    - $\forall x \in \mathbb{R},\ x \leq x$
    - $\forall x, y, z \in \mathbb{R},\ x \leq y$ and $y \leq z \implies x \leq z$
    - $\forall x, y \in \mathbb{R},\ x \leq y$ and $y \leq x \implies x = y$
- Arithmetic of the reals
  - Define the sum and product of two reals
  - Show they satisfy the properties of arithmetic
  - Show that the rules of arithmetic are equivalent to that of the rations
  - Proofs are not so easy and not so interesting

Real Arithmetic

- Given $x, y \in \mathbb{R}$, its sum is denoted $x + y \in \mathbb{R}$
- Given $x, y \in \mathbb{R}$, its product is denoted $xy \in \mathbb{R}$
- The following properties hold:
  - $\forall x, y \in \mathbb{R},\ x + y = y + x$
  - $\forall x, y, z \in \mathbb{R},\ (x + y) + z = x + (y + z)$
  - $\exists\ 0 \in \mathbb{R}$ such that $\forall z \in \mathbb{R}, 0 + z = z$
  - $\forall x \in \mathbb{R},\ \exists - x \in \mathbb{R}$ such that$x + (-x) = 0$
  - $\forall x, y, \in \mathbb{R},\ xy = yx$
  - $\forall x, y, z \in \mathbb{R},\ (xy)z = x(yz)$
  - $\exists 1 \in \mathbb{R}$ such that $\forall x \in \mathbb{R},\ 1x = x$
  - $\forall x \in \mathbb{R}$, if $x \neq 0$, then $\exists x^{-1}$ such that $xx^{-1} = 1$
  - $x(y + z) = xy + xz$

Real Ordering Properties

- $\forall x, y \in \mathbb{R}$, exactly one of the following is true:
  - $x < y$
  - $x = y$
  - $x > y$
- $\forall x, y, z \in \mathbb{R}$, if $x < y$ and $y < z$, then $x < z$
- $\forall x, y, z \in \mathbb{R}$, if $x < y$, then $x + z < y + z$
- $\forall x, y \in \mathbb{R}$, if $x, y > 0$, then $xy > 0$

Proposition 1.1.8: Consequences of Arithmetic and Ordering Propeties

- $\forall x \in \mathbb{R}$, $0x = 0$

$$0x = (x - x)x = x^2 - x^2 = 0$$

- $\forall x \in \mathbb{R}$, $x > 0 \iff -x < 0$
- $\forall x, y, z \in \mathbb{R}$, if $x > 0$, then $y < z \implies xy < xz$
- $\forall x, y, z \in \mathbb{R}$, if $x < 0$, then $y < z \implies xy > xz$
- $\forall x \in \mathbb{R}$, $x \neq 0 \implies x^2 > 0$
- $\forall x, y \in \mathbb{R}$, $x < y \implies x^{-1} > y^{-1}$
- $\forall x, y \in \mathbb{R}$, if $x, y > 0$,

$$x < y \iff x^2 < y^2$$

- $\forall x, y, z, w \in \mathbb{R}$,

$$x \leq y \text{ and } z \leq w \implies x + z \leq y + w$$

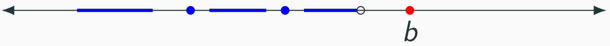
Upper Bound of a subset in $\mathbb{R}$

- An **upper bound** of a nonempty set $S \subset \mathbb{R}$ is a number $b \in \mathbb{R}$ such that

$$\forall s \in S,\ s \leq b$$

  - Example: 5 is an upper bound for $(-\infty, -1]$
- A set $S \subset \mathbb{R}$ is **bounded from above** if it has at least one upper bound
  - $\{2^{-n} \ :\ n \in \mathbb{N}\}$ is bounded from above by 1
  - $\mathbb{N}$ is not bounded from above

Least Upper Bound of a subset in $\mathbb{R}$



- The **least upper bound** of a set $S \subset \mathbb{R}$ that is bounded from below is an upper bound $m \in \mathbb{R}$ such that if $b \in \mathbb{R}$ is an upper bound, then

$$m \leq b$$

- The least upper bound of S need not be in S
- Example: For each $r \in \mathbb{R}$,

$$\sup(-\infty, r) = \sup(-\infty, r] = r$$

- For any cut $r = (A|B)$, $r \in \mathbb{R}$ is the least upper bound of $B \subset \mathbb{R}$

Completeness of the Reals

Theorem. *Any subset of* $\mathbb{R}$ *that is bounded from above has a least upper bound*

- This is the fundamental property that the reals have but the rationals do not
- It is a mathematically precise way to say that the real line has no holes

$\sqrt{2} \in \mathbb{R}$ Part 1

- The set $A = \{r \in \mathbb{R} \ : \ r^2 < 2\}$ is bounded from above
- If $r \in A$, then
  - $2 - r^2 > 0$
  - Therefore, there exists $0 < \epsilon < 1$ such that $2 - r^2 > \epsilon$
  - If $\delta > 0$ and $x = r + \delta$, then

$$2 - x^2 = 2 - r^2 - 2r\delta - \delta^2 > \epsilon - \delta(2r + \delta)$$

  - If $\delta \leq 1$, then $2r + \delta \leq 2r + 1$ and therefore

$$2 - x^2 > \epsilon - \delta(2r + \delta) > \epsilon - \delta(2r + 1)$$

  - It follows that if

$$\delta < \min\left(1, \frac{\epsilon}{2r + 1}\right),$$

    then

$$2 - x^2 > 0, \text{ i.e., } x^2 < 2$$

  - Therefore, $x \in A$ and $x > r$, which means r is not an upper bound of A

$\sqrt{2} \in \mathbb{R}$ **Part 2**

- If $r \in \mathbb{R}$ satisfies $r > 0$ and $r^2 > 2$, then r is an upper bound of A
  - If $s \in A$ and $s > 0$, then $s^2 < 2 < r^2$, which implies that $s < r$
- Let $\epsilon = r^2 - 2$
- If $x = r - \delta$, where $0 < \delta < r$, then

$$x^2 - 2 = (r - \delta)^2 - 2 = \epsilon - 2\delta r + \delta^2 > \epsilon - 2r\delta$$

- Therefore, if

$$\delta < \min\left(r, \frac{\epsilon}{2r}\right),$$

  then $x^2 > 2$
- It follows that x is an upper bound and $x < r$ and therefore r is not the least upper bound
- It follows that $m = \sup A$ satisfies neither $m^2 < 2$ nor $m^2 > 2$
- Therefore, $m^2 = 2$

Upper and Lower Bounds

- A nonempty set $S \subset \mathbb{R}$ is **bounded from above** if there exists $u \in \mathbb{R}$ such that

$$\forall s \in S, \ s \leq u$$

  - The **least upper bound** of S is called the **supremum** and denoted $\sup(S)$
- A nonempty set $S \subset \mathbb{R}$ is **bounded from below** if there exists $\ell \in \mathbb{R}$ such that

$$\forall s \in S, \ s \geq \ell$$

  - The **greatest lower bound** of S is called the **infimum** and denoted $\inf(S)$
- If a nonempty set $S \subset \mathbb{R}$ is bounded from both above and below, it is called **bounded**