MATH-UA 325 Analysis I Fall 2023

Archimedean Property of $\mathbb{R}$
Supremum and Infimum of $S \subset \mathbb{R}$
Absolute Value and Distance
Bounded Functions
Intervals
Sequence of Reals

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1.2.2 Archimidean Property

- If $x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ such that $n > x$
- Equivalently, N is not bounded from above
- Consequences
  - For any $x > 0$ and $y \in \mathbb{R}$, $\exists n \in \mathbb{N}$ such that $nx > y$
  - If $x, y \in \mathbb{R}$ and $x < y$, then $\exists q \in \mathbb{Q}$ such that $x < q < y$

1.2.3 Using Supremum and Infimum

- Review Propositions 1.2.6, 1.2.7, 1.2.8
- **IGNORE** Definition 1.2.9
    - We will **NEVER** use ∞ or $-\infty$ as a number
- Valid uses of ∞

$$[a, \infty) = \{x \in \mathbb{R} \ : \ x \geq a\}$$
$$(a, \infty) = \{x \in \mathbb{R} \ : \ x > a\}$$
$$(-\infty, a] = \{x \in \mathbb{R} \ : \ x \leq a\}$$
$$(-\infty, a) = \{x \in \mathbb{R} \ : \ x < a\}$$

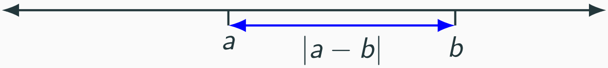
- Invalid uses of ∞

$$x = \infty$$
$$x \leq \infty$$
$$x \geq \infty$$

Maximum and minimum of a Finite Set

- Any finite set S of real numbers is bounded
- The supremum is called the **maximum** and denoted $\max(S)$
- The infimum is called the **minimum** and denoted $\min(S)$
- The maximum and minimum of S are always members of S

1.3 Absolute Value



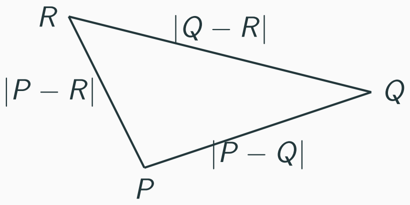
- The **absolute value** of $x \in \mathbb{R}$ is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

- Review Proposition 1.3.1 (basic properties of absolute value)
- Geometric definition of absolute value

$$|a - b| = \text{distance between } a \text{ and } b$$

1.3 Triangle Inequality in the Plane

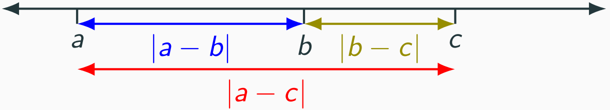


- Given two points A and B in the plane, let $|A - B|$ be the distance from A to B
- Given a triangle PQR,

$$|P - Q| + |Q - R| \geq |P - R|$$

- Equality holds iff P, Q, R lie on a line

1.3 Triangle Inequality in the Line



- $$
\begin{aligned}
|a-c| &\le |a-b| + |b-c| \\
|a-b| &\le |a-c| + |c-b| \\
|b-c| &\le |b-a| + |a-c| \\
|a-b| &\le |a| + |b| \\
|a+b| &\le |a| + |b|
\end{aligned}
$$

- If $|x| \le |y|$, then

$$|x| - |y| \le |x-y| = |y-x| \le |y| - |x|$$

1.3 Example of Bounded Function

- Find $M \in \mathbb{R}$ such that

$$\forall -1 \leq x \leq 5, \; -M \leq x^2 - 9x + 1 \leq M$$

- Since

$$\begin{aligned} |x^2 - 9x + 1| &\leq |x^2| + |9x| + 1 \\ &\leq 25 + 45 + 1 = 71, \end{aligned}$$

$M = 71$ is a valid solution

1.3 Bounded Functions

- Given a domain D and a function $f : D \to \mathbb{R}$, f is **bounded** if there exists $M \in \mathbb{R}$ such that

$$\forall x \in D,\ |f(x)| \leq M$$

- If $D = [-1, 5]$ and $f(x) = x^2 - 9x + 1$ for each $x \in D$, then f is bounded
- If $D = \mathbb{R}$ and $f(x) = x^2 - 9x + 1$ for each $x \in \mathbb{R}$, then f is unbounded
- If $D = \mathbb{R}$ and $f(x) = \text{arctan}$ for each $x \in \mathbb{R}$, then f is bounded

1.3 Supremum and Infimum of a Function

- A function $f : D \to \mathbb{R}$ is **bounded from above** if $f(D) \subset \mathbb{R}$ is bounded from above
  - The **supremum** of f is defined to be

$$\sup_{x \in D} f(x) = \sup f(D)$$

- A function $f : D \to \mathbb{R}$ is **bounded from below** if $f(D) \subset \mathbb{R}$ is bounded from below
  - The **infimum** of f is defined to be

$$\inf_{x \in D} f(x) = \inf f(D)$$

- If f and g are functions on a domain D and

$$\forall x \in \mathbb{D},\ f(x) \leq g(x),$$

then

$$\sup_{x \in D} f(x) \leq \sup_{x \in D} g(x) \text{ and } \inf_{x \in D} f(x) \leq \inf_{x \in D} g(x)$$

1.4 Intervals

- Given $a \leq b$,

$$
\begin{aligned}
(a, b) &= \{x \in \mathbb{R} \ : \ a < x < b\} \\
[a, b] &= \{x \in \mathbb{R} \ : \ a \leq x \leq b\} \\
[a, b) &= \{x \in \mathbb{R} \ : \ a \leq x < b\} \\
(a, b] &= \{x \in \mathbb{R} \ : \ a < x \leq b\} \\
(a, \infty) &= \{x \in \mathbb{R} \ : \ x > a\} \\
[a, \infty) &= \{x \in \mathbb{R} \ : \ x \geq a\} \\
(-\infty, b) &= \{x \in \mathbb{R} \ : \ x < b\} \\
[-\infty, b) &= \{x \in \mathbb{R} \ : \ x \leq b\} \\
(-\infty, \infty) &= \mathbb{R}
\end{aligned}
$$

1.4 Interval Types

- $(a, b), (a, b], [a, b), [a, b]$ are **bounded intervals**
- $(a, \infty), [a, \infty), (-\infty, a), (-\infty, a]$ are **unbounded intervals**
- $(a, b), (a, \infty), (-\infty, a)$ are **open intervals**
- $[a, b], [a, \infty), (-\infty, a]$ are **closed intervals**

Proposition 1.4.1: Characterization of an Interval

A set $I \subset \mathbb{R}$ *is a nonempty interval if and only if for any* $a, b \in I$ *and* $c \in \mathbb{R}$ *such that* $a \leq c \leq b$, $c \in I$.