MATH-UA 325 Analysis I Fall 2023

Sequence of Reals
Limit of a Sequence

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2.1 Sequence of Reals

- A **sequence** of reals is a function $s : \mathbb{N} \to \mathbb{R}$.
- Examples
  - $\forall n \in \mathbb{N},\ s(n) = 1$
  - $\forall n \in \mathbb{N},\ s(n) = 2n - 1$
  - $\forall n \in \mathbb{N},\ s(n) = \dfrac{1}{n}$
- Notation:
  - Instead of denoting the n-th element by $s(n)$, we will denote it by s_n
  - The sequence itself will also be denoted by simply s_n
  - The meaning of the notation s_n on its own is ambiguous
    - Use the context to determine whether it represents the n-element of the sequence or the entire sequence itself

2.1 Bounded Sequence of Reals

- A sequence is **bounded** if it is a bounded function, i.e.,

$$\exists B \in \mathbb{R} \text{such that} \forall n \in \mathbb{N},\ |s_n| \leq B$$

2.1 Limit of a Sequence

- We say that $a, b \in \mathbb{R}$ are equal within an **error tolerance** $\epsilon > 0$ if the distance between a and b is less than ϵ
    - $d(a, b) < \epsilon$
    - $|a - b| < \epsilon$
    - $a - \epsilon < b < a + \epsilon$
- A sequence $s_1, s_2, \cdots \in \mathbb{R}$ has a **limit** $L \in \mathbb{R}$ if the following holds:
    - For any error tolerance $\epsilon > 0$, only finitely many elements in the sequence are not equal to L within the error tolerance ϵ
    - Equivalently, the sequence $s_1, s_2, \ldots$ has a limit L if for any error tolerance $\epsilon > 0$, there exists $N_\epsilon > 0$ such that for every $n > N_\epsilon$, s_n is equal to L with the error tolerance ϵ
    - Equivalently, the sequence $s_1, s_2, \ldots$ has a limit L if

$$\forall \epsilon > 0,\ \exists N_\epsilon \in \mathbb{N} \text{ such that } \forall n > N_\epsilon,\ d(s_n, L) < \epsilon.$$

Examples

- The sequence

  $$s_n = 1, \ \forall n \in \mathbb{N}$$

  has the limit 1
- The sequence

  $$s_n = n, \ \forall n \in \mathbb{N}$$

  has no limit
- The sequence

  $$s_n = (-1)^n, \ \forall n \in \mathbb{N}$$

  has no limit

Example of a Sequence With a Limit

- The sequence

$$s_n = \frac{1}{n},\ \forall n \mathbb{N}$$

has the limit 0

  - If $\epsilon = \dfrac{1}{100}$, then for any $n > 100$,

$$d(s_n, 0) = d\left(\frac{1}{n}, 0\right) = \frac{1}{n} < \frac{1}{100}$$

  - For any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$\frac{1}{N_\epsilon} < \epsilon$$

  - For any $n > N_\epsilon$,

$$d(s_n, 0) = \frac{1}{n} < \frac{1}{N_\epsilon} < \epsilon$$

Notation and Terminology

- If a sequence has a limit, it is called a **convergent** sequence
- If $s_1, s_2, \ldots$ is a sequence with a limit L, we say that it **converges** to the limit L