MATH-UA 325 Analysis I
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Sequences
Monotone Sequences
Subsequences

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2.1 Sequence of Reals

- A **sequence** of reals is a function $s : \mathbb{N} \to \mathbb{R}$.
- Examples
  - $\forall n \in \mathbb{N},\ s(n) = 1$ is the sequence
    $$1, 1, 1, \ldots$$
  - $\forall n \in \mathbb{N},\ s(n) = 2n - 1$ is the sequence
    $$1, 3, 5, 7, \ldots$$
  - $\forall n \in \mathbb{N},\ s(n) = \frac{1}{n}$ is the sequence
    $$1, \frac{1}{2}, \frac{1}{3}, \ldots,$$

Notation for a Sequence

- Instead of denoting the n-th element by $s(n)$, we will denote it by s_n
- The notation for a sequence $s_1, s_2, \dots$ will be denoted

$$(s_n \ : \ n \geq 1)$$

  - It is assumed that the values of n are integers
  - The variable n can be replaced by any other one

$$(s_j \ : \ j \geq 1) = (s_n \ : \ n \geq 1)$$

  - The sequence need not start with $n = 1$
  - A sequence of the form $(s_{-2}, s_{-1}, s_0, \dots)$ is written

$$(s_k \ : \ k \geq -2)$$

  - This notation is not commonly used
- We use $(\cdots)$ instead of $\{\cdots\}$, because the order of the elements matters

2.1 Bounded Sequence of Reals

- A sequence $(s_n \ : \ n \geq n_0)$ is **bounded** if it is a bounded function, i.e.,

$$\exists B \in \mathbb{R} \text{ such that } \forall n \geq n_0, \ |s_n| \leq B$$

2.1 Limit of a Sequence

- We say that $a, b \in \mathbb{R}$ are equal within an **error tolerance** $\epsilon > 0$ if the distance between a and b is less than ϵ, i.e.,
    - $d(a, b) < \epsilon$
    - $|a - b| < \epsilon$
    - $a - \epsilon < b < a + \epsilon$
    - $b - \epsilon < a < b + \epsilon$
- A sequence $s_1, s_2, \cdots \in \mathbb{R}$ has a **limit** $L \in \mathbb{R}$ if the following holds:
    - For any error tolerance $\epsilon > 0$ (no matter how small), only finitely many elements in the sequence are not equal to L within the error tolerance ϵ
    - Equivalently, the sequence $s_1, s_2, \ldots$ has a limit L if

$$\forall \epsilon > 0,\ \exists N_\epsilon \in \mathbb{N} \text{ such that } \forall n > N_\epsilon,\ d(s_n, L) < \epsilon.$$

Examples

- The sequence
$$s_n = 1, \ \forall n \in \mathbb{N}$$
has the limit 1
- The sequence
$$s_n = n, \ \forall n \in \mathbb{N}$$
has no limit
- The sequence
$$s_n = (-1)^n, \ \forall n \in \mathbb{N}$$
has no limit

Example of a Sequence With a Limit

- The sequence

$$s_n = \frac{1}{n}, \ \forall n \in \mathbb{N}$$

has the limit 0

  - If $\epsilon = \dfrac{1}{100}$, then for any $n > 100$,

$$d(s_n, 0) = d\left(\frac{1}{n}, 0\right) = \frac{1}{n} < \frac{1}{100}$$

  - For any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$\frac{1}{N_\epsilon} < \epsilon$$

  - For any $n > N_\epsilon$,

$$d(s_n, 0) = \frac{1}{n} < \frac{1}{N_\epsilon} < \epsilon$$

Example

- Consider the sequence

$$s = \left(\frac{n^2+1}{n^2-n} : \; n \geq 2 \right)$$

- Since

$$s_n = \frac{n^2+1}{n^2-n} = \frac{1+\frac{1}{n^2}}{1-\frac{1}{n}},$$

  we know from calculus the limit has to be 1

- But we need to prove this

Proof

- First, we calculate the error and look for an upper bound of the error
- For each $n > 2$,

$$|s_n - 1| = \left|\frac{1+\frac{1}{n^2}}{1-\frac{1}{n}} - 1\right| = \left|\frac{1+\frac{1}{n^2}-(1-\frac{1}{n})}{1-\frac{1}{n}}\right| = \left|\frac{\frac{1}{n^2}+\frac{1}{n}}{1-\frac{1}{n}}\right|$$

$$= \frac{1}{n}\left(\frac{1+\frac{1}{n}}{1-\frac{1}{n}}\right) \leq \frac{1}{n}\left(\frac{2}{\frac{1}{2}}\right) = \frac{4}{n}$$

- Therefore, if for any $\epsilon > 0$, we choose N_ϵ such that

$$N_\epsilon > 2 \text{ and } \frac{4}{N_\epsilon} < \epsilon,$$

then for any $n > N_\epsilon$,

$$|s_n - 1| \leq \frac{4}{n} < \frac{4}{N_\epsilon} < \epsilon$$

- This implies that $\lim_{n\to\infty} s_n = 1$

Sequence Converges $\iff$ Errors Converge To Zero

- Given a sequence $s = (s_n : \ n \in \mathbb{N})$ and $L \in \mathbb{R}$, let

$$e_n = |s_n - L|$$

- If the limit of the sequence s is L, then for any $\epsilon > 0$, there exists $N_\epsilon > 0$ such that for any $n > N_\epsilon$,

$$e_n = |s_n - L| < \epsilon$$

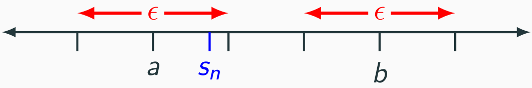
  and therefore $\lim_{n\to\infty} e_n = 0$

- Conversely, if $\lim_{n\to\infty} e_n = 0$, then for any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that if $n > N_\epsilon$, then

$$|s_n - 1| = |e_n - 0| < \epsilon,$$

  which implies that $\lim_{n\to\infty} s_n = L$

A Convergent Sequence Has a Unique Limit



- Suppose $a = \lim_{n \to \infty} s_n$ and $b \neq a$
- Idea: If s_n is close to a, it cannot be close to b
- Let $\epsilon = \frac{|a-b|}{3}$
- All but finitely many numbers in the sequence are in the interval $(a - \epsilon, a + \epsilon)$, i.e., only finitely many numbers are outside this interval
- Since

  $$(a - \epsilon, a + \epsilon) \cap (b - \epsilon, b + \epsilon) = \emptyset,$$

  it follows that only finitely many numbers in the sequence lie in $(b - \epsilon, b + \epsilon)$
- Therefore, b cannot also be a limit of the sequence

Notation and Terminology

- If a sequence has a limit, it is called a **convergent** sequence
- If $s_1, s_2, \ldots$ is a sequence with a limit L, we say that it **converges** to the limit L
- The limit of a convergent sequence $(s_n \ : \ n \geq n_0)$ is written as

$$\lim_{n \to \infty} s_n$$

A Convergent Sequence is Bounded

- A sequence is **bounded** if there exists $M > 0$ such that

$$\forall n \in \mathbb{N},\ d(x_n, x_1) < M$$

- Suppose

$$\lim_{n\to\infty} s_n = L$$

- There exists $N \in \mathbb{N}$ such that

$$\forall n > N,\ |s_n - L| < 1$$

  and therefore

$$|s_n| = |s_n - L + L| \leq |s_n - L| + |L| < |L| + 1$$

- Let

$$m = \max(|s_1|, \ldots, |s_N|)$$

- It follows that if $M = \max(|L| + 1, m + 1)$, then

$$\forall n \in N,\ |s_n| < M$$

Example of Convergent Sequence

- Consider the sequence

$$s_n = \frac{n^2+1}{n^2-n}, \;\; n \geq 2$$

- Observe that

$$s_n = \frac{n^2+1}{n^2-n} = \frac{n^2(1+\frac{1}{n^2})}{n^2(1-\frac{1}{n})} = \frac{1+\frac{1}{n^2}}{1-\frac{1}{n}}$$

  and therefore, if $n > 1$, then

$$s_n - 1 = \frac{1}{n}\left(\frac{1+\frac{1}{n}}{1-\frac{1}{n}}\right)$$

- From this, we can see that

$$\lim_{n\to\infty} s_n = 1$$

Monotone Sequences in $\mathbb{R}$

- A sequence s_n is **monotone increasing** if for any $n \in \mathbb{N}$,

$$s_n \leq s_{n+1}.$$

  - Example:

$$s_n = n, n \in \mathbb{N}$$

- A sequence s_n is **monotone decreasing** if for any $n \in \mathbb{N}$,

$$s_n \geq s_{n+1}.$$

  - Example:

$$s_n = \frac{1}{n},\ \mathbb{N}$$

- A sequence s_n is **monotone** if it is monotone increasing or monotone decreasing

Convergence of Monotone Sequence

- A monotone sequence converges if and only if it is bounded
- If a monotone increasing sequences s_n is bounded, then

$$\lim_{n \to \infty} s_n = \sup s_n$$

- If a monotone decreasing sequences s_n is bounded, then

$$\lim_{n \to \infty} s_n = \inf s_n$$

Bounded Monotone Increasing Sequence Converges

- Let s_n be a bounded monotone sequence and $s = \sup s_n$
- For any $\epsilon > 0$, $s - \epsilon$ is not an upper bound
- Therefore, there exists $N_\epsilon \in \mathbb{N}$ such that

$$s_{N_\epsilon} > s - \epsilon$$

- Since s_n is monotone increasing,

$$\forall n \geq N_\epsilon,\ s_n \geq s_{N_\epsilon} > s - \epsilon$$

which implies that

$$|s - s_n| < \epsilon$$

- Therefore, for any $\epsilon > 0$, there exists N_ϵ such that

$$n > N_\epsilon \implies |s - s_n|,$$

i.e.,

$$\lim_{n \to \infty} s_n = s$$

Example

- Let
$$s_n = \frac{n-2}{n},\ \forall n \in \mathbb{N}$$
- s_n is bounded and monotone increasing
$$s_n = \frac{n-2}{n} = 1 - \frac{2}{n} < 1$$
- $$\lim_{n\to\infty} s_n = 1$$
  because for any $\epsilon > 0$, there exists $N > 0$ such that
$$\frac{2}{N} < \epsilon$$
  and therefore, for any $n > N$,
$$|1 - s_n| = \left|1 - (1 - \frac{2}{n})\right| = \frac{2}{n} < \frac{2}{N} < \epsilon$$

Subsequences

- Let s_n, $n \in \mathbb{N}$, be a sequence
- Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function, i.e., a strictly increasing sequence
- Denote

$$n_i = f(i)$$

- $t_i = s_{n_i},\ i \in \mathbb{N}$ is a subsequence
- Example: If $n_i = 2i - 1$, then we get the subsequence

$$s_1, s_3, s_5, \ldots$$

- Example: Consider the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \cdots, \text{ i.e., } \forall n \in \mathbb{N},\ s_n = \frac{1}{n}$$

  - A subsequence is

$$1, \frac{1}{4}, \frac{1}{9}, \ldots, \text{ i.e., } \forall i \in \mathbb{N},\ t_i = s_{i^2} = \frac{1}{i^2}$$

Facts About Subsequences

- If a sequence converges to L, then any subsequence converges to L
- If a subsequence converges to L, then the sequence need not be convergent
  - Example: A subsequence of

$$0, 1, 0, 1, \cdots$$

is

$$1, 1, 1, \ldots$$