MATH-UA 325 Analysis I
Fall 2023

Subsequences
Squeeze Lemma
Algebra of Convergent Sequences

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2.2.1 Lemma

- If $\forall n \in \mathbb{N},\ a_n \leq b_n$, then

$$\lim_{n\to} a_n \leq \lim_{n\to} b_n$$

- Proof:
  - Let

$$a = \lim_{n\to\infty} a_n \text{ and } b = \lim_{n\to\infty} b_n$$

  - For any $\epsilon > 0$, there exists $n_\epsilon \mathbb{N}$ such that

$$\max(|a - a_n|, |b - b_n|) < \epsilon,$$

which implies

$$-\epsilon < (a - a_n), (b - b_n) < \epsilon$$

which implies

$$b-a = (b-b_n)-(a-a_n)+(b_n-a_n) \geq (b-b_n)-(a-a_n) > -2\epsilon$$

  - Since this holds for any $\epsilon > 0$, it follows that

$$b \geq a$$

Lemma

- Given a sequence $(a_n : \; n \geq n_0)$ and $a \in \mathbb{R}$, if there is a sequence $(e_n : \; n \geq n_0)$ that satisfies for each $n \geq n_0$,

$$|a_n - a| \leq e_n \text{ and } \lim_{n \to \infty} e_n = 0,$$

then

$$\lim_{n \to \infty} a_n = a$$

- Proof:
  - For any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$n > N_\epsilon \implies e_n < \epsilon,$$

which implies

$$n > N_\epsilon \implies |a_n - a| < e_n < \epsilon$$

  - Therefore, $\lim_{n \to \infty} a_n = a$

Squeeze Lemma

- If $(a_n : \ n \geq n_0)$, $(b_n : \ n \geq n_0)$, $(x_n : \ n \geq n_0)$ are sequences such that
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \text{ and } \forall n \geq n_0,\ a_n \leq x_n \leq b_n,$$
then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} b_n$
- Sketch of proof:
  - Show that the sequence $(e_n : \ n \geq n_0)$, where $e_n = b_n - a_n$ converges to 0
  - Since $|x_n - a_n| = x_n - a_n < b_n - a_n = e_n$, it follows by the previous lemma that
  $$\lim_{n\to\infty} |x_n - a_n| = 0$$
  - Show that this implies, by the definition of a convergent sequence, that
  $$\lim_{n\to\infty} x_n = \lim_{n\to\infty} a_n$$

Addition of Convergent Sequences

- If $(s_k \;:\; k \geq 1)$ and $(t_k \;:\; k \geq 1)$ are convergent sequences, then

$$\lim_{k\to\infty}(s_k + t_k) = (\lim_{k\to\infty} s_k) + (\lim_{k\to\infty} t_k)$$

Proof (Part 1)

- Suppose

$$s = \lim_{k\to\infty} s_k \text{ and } t = \lim_{k\to\infty} t_k$$

- For any $\epsilon_1 > 0$, there exists N_1 such that

$$n > N_1 \implies d(s_n, s) < \epsilon_1$$

and For any $\epsilon_2 > 0$, there exists N_2 such that

$$n > N_2 \implies d(t_n, s) < \epsilon_1$$

- It follows that if $n > \max(N_1, N_2)$,

$$\begin{aligned} |(s+t) - (s_n + t_n)| &= |(s - s_n) + (t - t_n)| \\ &\leq |s - s_n| + |t - t_n| \\ &< \epsilon_1 + \epsilon_2 \end{aligned}$$

Proof (Part 2)

- For any $\epsilon > 0$, let $\epsilon_1, \epsilon_2 > 0$ satisfy

$$\epsilon_1 + \epsilon_2 = \epsilon$$

- Let N_1, N_2 be as defined in previous slide

$$N = \max(N_1, N_2)$$

- Then for any $n > N$,

$$|(s + t) - (s_n + t_n)| < \epsilon$$

Addition, Multiplication, Division of Sequences

- If $(s_k \ : \ k \geq 1)$ and $(t_k \ : \ k \geq 1)$ are convergent sequences, then

$$\lim_{k\to\infty}(s_k + t_k) = (\lim_{k\to\infty} s_k) + (\lim_{k\to\infty} t_k)$$
$$\lim_{k\to\infty}(s_k t_k) = (\lim_{k\to\infty} s_k)(\lim_{k\to\infty} t_k)$$

- If, in addition,

$$(\lim_{k\to\infty} t_k) \neq 0,$$

then

$$\lim_{k\to\infty}\left(\frac{s_k}{t_k}\right) = \frac{\lim_{k\to\infty} s_k}{\lim_{k\to\infty} t_k}$$

Product of Convergent Sequences is Convergent (Part 1)

- Suppose that

$$\lim_{n\to\infty} a_n = A \text{ and } \lim_{n\to\infty} b_n = B$$

- Let $c_n = a_n b_n$
- We want to show that $(c_n : \ n \in \mathbb{N})$ converges to AB
- We need to estimate the error $|c_n - AB|$ in terms of the errors $|a_n - A|$ and $|b_n - B|$
- The formula for the error is

$$\begin{aligned} |c_n - AB| &= |a_n b_n - AB| \\ &= |(a_n - A)b_n + Ab_n - AB| \\ &= |(a_n - A)b_n + A(b_n - B)| \\ &\leq |a_n - A||b_n| + |A||b_n - B| \end{aligned}$$

Product of Convergent Sequences is Convergent (Part 2)

- The formula for the error is

$$|c_n - AB| \leq |b_n||a_n - A| + |A||b_n - B|$$

- Since $(a_n : \ n \in \mathbb{N})$ and $(b_n : \ n \in \mathbb{N})$ are convergent, they are bounded and therefore there exist $M > 0$ such that

$$\forall n \in \mathbb{N}, \ |a_n|, |b_n| < M$$

- Therefore, for all $n \in \mathbb{N}$,

$$|c_n - AB| \leq M|a_n - A| + M|b_n - B|$$

- For any $\epsilon_1 > 0$, there exists N_1 such that for each $n > N_1$,

$$|a_n - A| < \epsilon_1$$

- For any $\epsilon_2 > 0$, there exists N_2 such that for each $n > N_2$,

$$|b_n - B| < \epsilon_2$$

Product of Convergent Sequences is Convergent (Part 3)

- For any $\epsilon_1, \epsilon_2 > 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$n > \max(N_1, N_2) \implies |a_n - A| < \epsilon_1 \text{ and } |b_n - B| < \epsilon_2$$

and therefore,

$$|c_n - AB| \leq M(\epsilon_1 + \epsilon_2)$$

- For any $\epsilon > 0$, let

$$\epsilon_1 = \epsilon_2 = \frac{\epsilon}{2M}$$

- We have shown that for any $\epsilon > 0$, there ecists $N_\epsilon \in \mathbb{N}$ such that

$$n > \mathbb{N}_\epsilon \implies |c_n - AB| \leq \epsilon$$

- I.e.,

$$\lim_{n\to\infty} a_n b_n = AB$$

Reciprocal of Convergent Sequence is Convergent (Part 1)

- Suppose $s_n \neq 0$ for all $n\mathbb{N}$ and

$$\lim_{n\to\infty} s_n = L \neq 0$$

- We want to prove that

$$\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{L} > 0$$

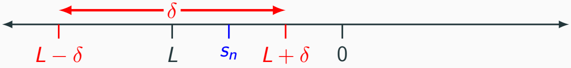
- The error is

$$\left|\frac{1}{s_n} - \frac{1}{L}\right| = \left|\frac{L - s_n}{s_n L}\right| = \left|\frac{s_n - L}{s_n L}\right|$$

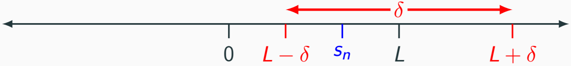
- We need an upper bound for $\dfrac{1}{|s_n L|}$
  - I.e., a positive lower bound for $|s_n|$

Reciprocal of Convergent Sequence is Convergent (Part 2)

- If $L < 0$,



- If $L > 0$,



- Therefore, if $|s_n - L| < \delta$, then

$$|s_n| = |L + s_n - L| \geq |L| - |s_n - L| > |L| - \delta$$

Reciprocal of Convergent Sequence is Convergent (Part 3)

- For any $\delta > 0$, there exists $N_\delta > 0$ such that

$$\forall n > N_\delta,\ |s_n - L| < \delta$$

- If $\delta < \frac{L}{2}$, then

$$|s_n| > |L| - \delta > \frac{|L|}{2}$$

- Therefore, if $n > N_\delta$, then

$$\left|\frac{1}{s_n} - \frac{1}{L}\right| = \frac{|s_n - L|}{|s_n||L|} < \frac{2}{L^2}|s_n - L| < \frac{2}{L^2}\delta$$

Reciprocal of Convergent Sequence is Convergent (Part 4)

- For any $\epsilon > 0$, let $\delta = \dfrac{L^2}{2}\epsilon$ and $M_\epsilon = N_\delta$
- It follows that for any $n > M_\epsilon$

$$\left|\frac{1}{s_n} - \frac{1}{L}\right| = \frac{|s_n - L|}{|s_n L|} < \frac{2}{L^2}\delta = \epsilon$$

- Therefore,

$$\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{\lim_{n\to\infty} s_n}$$

Square Root of Convergent Sequence is Convergent

- Suppose $(s_n : \; n \geq \mathbb{N})$ is a convergent sequence such that $\forall n \in \mathbb{N}, \; s_n \geq 0$ and

$$\lim_{n \to \infty} s_n = L$$

- We want to show that $\lim_{n \to \infty} \sqrt{s_n} = \sqrt{L}$
- The error is

$$\begin{aligned} |\sqrt{s_n} - \sqrt{L}| &= \left| \frac{(\sqrt{s_n} - \sqrt{L})(\sqrt{s_n} + \sqrt{L})}{\sqrt{s_n} + \sqrt{L}} \right| \\ &= \frac{|s_n - L|}{\sqrt{s_n} + \sqrt{L}} \end{aligned}$$

- The rest of the proof is a homework exercise