MATH-UA 325 Analysis I
Fall 2023

Square Root of Convergent Sequence
Convergence Tests
Geometric Sequences

Deane Yang

Updated September 28, 2023

Courant Institute of Mathematical Sciences
New York University

Square Root of Convergent Sequence is Convergent

- Suppose $(s_n : \ n \geq \mathbb{N})$ is a convergent sequence such that $\forall n \in \mathbb{N}, \ s_n \geq 0$ and
$$\lim_{n\to\infty} s_n = L$$
- We want to show that $\lim_{n\to\infty} \sqrt{s_n} = \sqrt{L}$
- The error is
$$\begin{aligned} |\sqrt{s_n} - \sqrt{L}| &= \left| \frac{(\sqrt{s_n} - \sqrt{L})(\sqrt{s_n} + \sqrt{L})}{\sqrt{s_n} + \sqrt{L}} \right| \\ &= \frac{|s_n - L|}{\sqrt{s_n} + \sqrt{L}} \end{aligned}$$
- The rest of the proof is a homework exercise

Convergence Tests

- Consider a sequence $(x_n : \ n \geq n_0)$
- If $x \in \mathbb{R}$ and $(e_n : \ n \geq n_0)$ satisfy

$$\lim_{n\to\infty} e_n = 0 \text{ and } \forall n \geq n_0,\ |x - x_n| \leq e_n,$$

  then

$$\lim_{n\to\infty} x_n = x$$

- Observe that $a_n \geq 0$ for all $n \in \mathbb{N}$
- For any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$\forall n > N_\epsilon,\ |x_n - x| \leq a_n = |a_n - 0| < \epsilon$$

Geometric Sequences (Part 1)

- Given $c > 0$, consider the sequence

$$(c^n : \; n \geq n \in \mathbb{N})$$

- If $c = 1$, then $\lim_{n\to\infty} c^n = 1$
- If $c < 1$, then $c^n > c^{n+1}$
  - The sequence is bounded and decreasing and therefore convergent
  - If $\lim_{n\to\infty} c^n = x$, then on one hand,

    $$\lim_{n\to\infty} c^{n+1} = x$$

    but on the other hand,

    $$\lim_{n\to\infty} c^{n+1} = \lim_{n\to\infty} cc^n = c \lim_{n\to\infty} = cx$$

  - Therefore, $0 = x - cx = (1-c)x$, which, since $1 - c \neq 0$ implies that $x = 0$

Geometric Sequences (Part 2)

- If $c > 1$, then $c^{-1} < 1$ and therefore

$$\lim_{n\to\infty} (c^{-1})^n = 0$$

- For any $B > 0$, there exists $N > 0$ such that

$$c^{-n} < \frac{1}{B}, \text{ i.e., } B < c^n$$

- It follows that the sequence $(c^n : \; n \geq 1)$ is unbounded
- It follows that the sequence $(c^n : \; n \geq 1)$ is divergent

Ratio Test for Sequences

- Let $(s_n : \ n \geq n_0)$ be a sequence
- If there exists $0 < r < 1$ and $N \in \mathbb{N}$ such that
$$\forall n \geq N, \ \frac{|s_{n+1}|}{|s_n|} \leq r,$$
then $\lim_{n\to\infty} s_n = 0$
- If there exists $r > 1$ and $N \in \mathbb{N}$ such that
$$\forall n \geq N, \ \frac{|s_{n+1}|}{|s_n|} \geq r,$$
then the sequence diverges
- If
$$\lim_{n\to\infty} \frac{|s_{n+1}|}{|s_n|} = 1,$$
then the sequence can diverge or converge

Proof When $r < 1$

- If $n > N$, then which implies

$$|s_n| = \frac{|s_n|}{|s_{n-1}|}\frac{|s_{n-1}|}{|s_{n-2}|}\cdots\frac{|s_{N+1}|}{|s_N|}|s_N| \le r^{n-N}|s_N|$$

- For any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$r^{N_\epsilon} < \frac{\epsilon}{|s_N|}$$

- Therefore, for any $n > N_\epsilon + N$,

$$|s_n| \le r^{n-N}|s_N| < r^{N_\epsilon}||s_N| < \epsilon$$

Proof When $r > 1$

- If $n > N$, then which implies

$$|s_n| = \frac{|s_n|}{|s_{n-1}|}\frac{|s_{n-1}|}{|s_{n-2}|}\cdots\frac{|s_{N+1}|}{|s_N|}|s_N| \geq r^{n-N}|s_N|$$

- For any $B > 0$, there exists $N_{1/B}$ such that for any $n > N_{1/B}$,

$$r^{-n} < \frac{1}{B}$$

  and therefore

$$r^n > B$$

- It follows that, for any $n > N_{1/B} + N$,

$$|s_n| \geq r^{n-N}|s_N| > r^{N_{1/B}}|s_N| > B|s_N|$$