MATH-UA 325 Analysis I
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Limit Supremum and Limit Infimum of Sequence

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Limit Superior of a Sequence

- Let $(s_n: \; n \geq n_0)$ be a bounded sequence
- For each $n \geq n_0$, let

$$a_n = \sup(s_k: \; k \geq n) \text{ and } b_n = \inf(s_k: \; k \geq n)$$

- For each $n \geq n_0$, since

$$(s_k: \; k \geq n+1) \subset (s_k : k \geq n),$$

  it follows that

$$a_n = \sup(s_k: \; k \geq n) \geq \sup(s_k: \; k \geq n+1) = a_{n+1}$$

  and therefore $(a_n: \; n \geq n_0)$ is a decreasing sequence
- $(a_n: \; n \geq n_0)$ is bounded, because any s_n is a lower bound
- The limit superior of $(s_n: \; n \geq n_0)$ is defined to be

$$\limsup_{n \to \infty} s_n = \lim_{n \to \infty} a_n$$

Limit Infimum of a Sequence

- Analogous story
- The sequence $(b_n : \ n \geq n_0$, where

$$b_n = \inf(s_k : \ k \geq n)$$

  is a bounded increasing sequence
- The limit infimum of $(s_n : \ n \geq n_0)$ is defined to be

$$\liminf_{n \to \infty} s_n = \lim_{n \to \infty} b_n$$

Examples

$$\liminf_{n\to\infty}((-1)^n : \; n \geq n_0) = -1$$

$$\limsup_{n\to\infty}((-1)^n : \; n \geq n_0) = 1$$

$$\liminf_{n\to\infty}\left(\frac{(-1)^n}{n} : \; n \geq n_0\right) = 0$$

$$\limsup_{n\to\infty}\left(\frac{(-1)^n}{n} : \; n \geq n_0\right) = 0$$

$\liminf = \limsup \iff$ **Convergence**

- If $(x_n : \; n \geq n_0)$ is a bounded sequence, then

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$$

- Proposition 2.3.5: *A bounded sequence* $(x_n : \; n \geq n_0)$ *converges if and only if*

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} s_n$$

Proof

- If $a_n = \sup(x_k : \ k \geq n)$ and $b_n = \inf(x_k : \ k \geq n)$, then for each $n \geq n_0$,

$$b_n \leq x_n \leq a_n$$

- By assumption,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$$

- By Squeeze Lemma, $(x_n : \ n \geq n_0)$ converges and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} x_n = \lim_{n\to\infty} a_n$$

$\liminf$ and $\limsup$ of a Subsequence

- Let $(x_{n_k} : \ k \geq k_0)$ be a subsequence of a sequence $(x_n : \ n \geq n_0)$
- For each $n \geq n_0$, let

$$a_n = \sup(x_k : \ k \geq n) \text{ and } a'_n = \sup(x_{n_j} : \ n_j \geq n)$$

- Since $(x_{n_j} : \ n_j \geq n) \subset (x_k : \ k \geq n)$, it follows that

$$a'_n \leq a_n$$

- Therefore,

$$\limsup_{k\to\infty} x_{n_k} = \lim_{n\to\infty} a'_n \leq \lim_{n\to\infty} a_n = \limsup_{n\to\infty} x_n$$

- It follows that

$$\liminf_{n\to\infty} x_n \leq \liminf_{k\to\infty} x_{n_k} \leq \limsup_{k\to\infty} x_{n_k} \leq \limsup_{n\to\infty} x_n$$