MATH-UA 325 Analysis I
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Bolzano-Weierstrass Theorem
Cauchy Sequences

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Bolzano-Weierstrass Theorem

Theorem
Any bounded sequence has a convergent subsequence.

Proof (Part 1)

- Let $(x_n : \ n \geq n_0)$ be a bounded sequence
- Let a_0 be a lower bound and b_0 an upper bound of the sequence
- Therefore,

$$(x_n \ : \ n_0) \subset [a_0, b_0]$$

- Let $c_0 = \dfrac{a_0 + b_0}{2}$
- At least one of the intervals $[a_0, c_0]$ and $[c_0, b_0]$ contains a subsequence
- If $[a_0, b_0]$ contains a subsequence, then let

$$a_1 = a_0, \ b_1 = c_0, \text{ and } x_{n_1} \in [a_1, b_1]$$

- Otherwise, let

$$a_1 = c_0, \ b_1 = b_0, \text{ and } x_{n_1} \in [a_1, b_1]$$

Proof (Part 2)

- Continue by induction
- Let $c_k = \dfrac{a_k + b_k}{2}$
- At least one of the intervals $[a_k, c_k]$ and $[c_k, b_k]$ contains a subsequence of $(x_n : \ n \geq n_k)$
- If $[a_k, b_k]$ contains a subsequence, then let

$$a_{k+1} = a_k, \ b_{k+1} = c_k, \text{ and } x_{n_{k+1}} \in [a_{k+1}, b_{k+1}]$$

- Otherwise, let

$$a_{k+1} = c_k, \ b_{k+1} = b_k, \text{ and } x_{n_{k+1}} \in [a_{k+1}, b_{k+1}]$$

- Observe that the sequences $(a_n : \ n \geq 0)$ and $(b_n : \ n \geq 0)$ are bounded, monotone and therefore convergent

Proof (Part 3)

- For each $n \geq 0$,

$$a_n \leq b_n \text{ and } b_n - a_n = \frac{b_0 - a_0}{2^n}$$

- Therefore, for each $n \geq 0$,

$$\lim_{n\to\infty} b_n - a_n \leq \lim_{n\to\infty} \frac{b_0 - a_0}{2^n} = 0$$

- Since for each $k \geq 0$,

$$a_{n_k} \leq x_{n_k} \leq b_{n_k} \text{ and } \lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} b_{n_k},$$

  it follows by the Squeeze Lemma that

$$\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

Cauchy Sequences

- Goal: Figure out whether a sequence converges even if you do not know what the limit is
- Original definition of convergence:

$$\lim_{n \to \infty} x_n = L,$$

  if for any $\epsilon > 0$, only finitely many elements in the sequence lie outside the interval $(L - \epsilon, L + \epsilon)$
- Cauchy criterion: A sequence $(x_n : \; n \geq n_0)$ is a **Cauchy sequence** if for any $\epsilon > 0$, there exists an open interval I of length ϵ such that only finitely many elements of the sequence lie outside I
- Equivalent definition: A sequence $(x_n : \; n \geq n_0)$ is a **Cauchy sequence** if for any $\epsilon > 0$, there exists $N_\epsilon > 0$ such that for any $n, m > N_\epsilon$,

$$|x_n - x_m| < \epsilon$$

Convergent $\implies$ Cauchy

- If
$$\lim_{n \to \infty} x_n = L,$$
then for any ϵ, there exists $N_{\epsilon/2} \geq n_0$ such that
$$\forall n > N_{\epsilon/2}, \ |x_n - L| < \frac{\epsilon}{2}$$
- Therefore, for any $n, m > N_{\epsilon/2}$,
$$|x_n - x_m| = |(x_n - L) + (L - x_m)| \leq |x_n - L| + |x_m - L| < \epsilon$$

Cauchy $\implies$ Bounded

- Let $(x_n : \; n \geq n_0)$ be a Cauchy sequence
- There exists $N \in \mathbb{N}$ such that for any $n, m > N$,

$$|x_n - x_m| < 1$$

- It follows that for any $n > N$,

$$|x_n - x_{N+1}| < 1$$

- It follows that for any $n \geq n_0$,

$$|x_n| \leq \max(|x_1|, \ldots, |x_N|, |x_{N+1}| + 1)$$

and therefore the sequence is bounded

Cauchy $\implies$ Convergent (Part 1)

- Let $a_n = \inf(x_k : \ k \geq n)$ and $b_n = \sup(x_k : \ k \geq n)$
- Let

$$a = \liminf_{n\to\infty} x_n = \lim_{n\to\infty} a_n$$

$$b = \limsup_{n\to\infty} x_n = \lim_{n\to\infty} b_n$$

- For any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that the following hold: $\forall n > N_\epsilon$,

$$a_n - a < \epsilon$$

$$b - b_n < \epsilon$$

- There exists $M_\epsilon > N_\epsilon$ such that for any $n, m \geq n_0$,

$$|x_n - x_m| < \epsilon$$

Cauchy $\implies$ Convergent (Part 2)

- Moreover, for each $n > M_\epsilon$, there exists $n_j, n_k > n$ such that

$$x_{n_j} - a_n < \epsilon \text{ and } b_n - x_{n_k} < \epsilon$$

- Therefore, for any $\epsilon > 0$,

$$\begin{aligned} |a - b| &= |a - a_n + a_n - x_{n_j} + x_{n_j} - x_{n_k} + x_{n_k} - b_n + b_n - b| \\ &\leq |a - a_n| + |x_{n_j} - a_n| + |x_{n_j} - x_{n_k}| + |x_{n_k} - b_n| + |b_n - b| \\ &\leq 5\epsilon \end{aligned}$$

- Therefore,

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$$

and the sequence converges