MATH-UA 325 Analysis I
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Completeness Via Cauchy Sequences
Open and Closed Sets in $\mathbb{R}$
Geometric Series

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Completeness Via Cauchy Sequences

- Completeness via supremum and infimum
  - Every subset $S \subset \mathbb{R}$ that is bounded from above has a least upper bound, denoted $\sup(S)$
  - Every subset $S \subset \mathbb{R}$ that is bounded from above has a greatest lower bound, denoted $\inf(S)$
- Completeness via Cauchy sequences
  - Every Cauchy sequence in $\mathbb{R}$ converges to a limit in $\mathbb{R}$

Cauchy Sequence Has Limit $\implies$ Bounded Set Has Supremum

- Proof is essentially same as Bolzano-Weierstrass
- Let $S \subset \mathbb{R}$ be a nonempty subset bounded from above
    - Assume S is infinite
- Construct two sequences $(a_n : n \geq 0)$ and $(b_n : n \geq 0)$ by induction
- $n = 0$: Let $a_0 \in S$ and b_0 be an upper bound of S
- Inductive assumption: $a_n \in S$ and b_n is an upper bound for S
    - Let $c_n = \frac{1}{2}(a_n + b_n)$
    - If c_n is an upper bound for S, then let $a_{n+1} = a_n$ and $b_{n+1} = c_n$
    - Otherwise, there exists $a_{n+1} \in S$ such that $a_{n+1} > c_n$, and let $b_{n+1} = b_n$

Cauchy Sequence Has Limit $\implies$ Bounded Set Has Supremum

- $(a_n : n \geq 0)$ is increasing, and $(b_n : n \geq 0)$ is decreasing
- $|b_n - a_n| = 2^{-n}|b_0 - a_0|$
- For any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$\forall n > N_\epsilon, |b_n - a_n| < \ 2^{-n}|b_0 - a_0| < \epsilon$$

- For any $j, k > N_\epsilon$ such that $j \leq k$,

$$a_j \leq a_k \leq b_k \leq b_j$$

  and therefore

$$|a_j - a_k| \leq |a_j - b_j| < \epsilon \text{ and } |b_j - b_k| < |a_j - b_j| < \epsilon$$

- This implies both sequences are Cauchy and therefore have limits

$$a = \lim_{n\to\infty} a_n \text{ and } b = \lim_{n\to\infty} b_n$$

Cauchy Sequence Has Limit $\implies$ Bounded Set Has Supremum

- For any $n \geq 0$,

$$b - a \leq b_n - a_n \leq (b_0 - a_0)2^{-n},$$

  which implies $a = b$
- b is an upper bound
  - For any $s \in S$ and $n \geq 0$,

$$s \leq b_n$$

  - Therefore,

$$\lim_{n \to \infty} s \leq \lim_{n \to \infty} b_n = b$$

- b is the least upper bound
  - Let $c < b$
  - Since $\lim_{n \to \infty} a_n = b$, there exists $n \in \mathbb{N}$ such that

$$c < a_n \leq b$$

  - Therefore, c is not an upper bound

Open and closed sets in $\mathbb{R}$

- A subset $S \subset \mathbb{R}$ is **closed** if every Cauchy sequence in S converges to a limit in S
  - Examples:
    - The empty set
    - A point in $\mathbb{R}$
    - $[a, b]$, $[a, \infty)$, $(-\infty, a]$ for any $a, b \in \mathbb{R}$
    - A finite union of closed sets
    - An infinite intersection of closed sets
- A subset $S \subset \mathbb{R}$ is **open** if for any $x \in S$, there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subset S$
  - Examples:
    - (a, b), (a, ∞), $(-\infty, a)$ for any $a, b \in \mathbb{R}$
    - An infinite union of open sets
    - A finite intersection of open sets
- A set $S \subset \mathbb{R}$ is open if and only if $\mathbb{R} \backslash S$ is closed

Series

- Given a sequence $(x_n : \; n \geq n_0)$, consider the infinite sum
$$\sum_{n=n_0}^{\infty} x_n = x_{n_0} + x_{n_0+1} + \cdots ,$$
which is also called a series
- The sequence of partial sums is $(s_n : n \geq n_0)$, where
$$s_n = x_{n_0} + \cdots + x_n = \sum_{k=n_0}^{n} x_k$$
- The infinite sum is defined to be **convergent** if the sequence of partial sums is convergent
    - If so, we say that
$$\sum_{n=n_0}^{\infty} x_n = \lim_{n \to \infty} s_n$$
- If the sequence of partial sums diverges, then the series is **divergent**

Tails of Sequences and Series

- A **tail** of a sequence $(s_n : n \geq n_0)$ is a sequence

$$(s_n : n \geq N),$$

where N is some integer greater than n_0

  - A sequence converges if and only if a tail does

- A **tail** of a series

$$\sum_{n=n_0}^{\infty} x_n$$

is a series

$$\sum_{n=N}^{\infty} x_n,$$

where N is some integer greater than n_0

  - A series converges if and only if a tail does

Geometric Series and Sum

- Geometric series: Given $r \in \mathbb{R}$, consider the series

$$1 + r + r^2 + \cdots = \sum_{k=0}^{\infty} r^k$$

- Basic algebraic formula:

$$1 - r^{N+1} = (1-r)(1 + r + r^2 + \cdots + r^N) = (1-r)\sum_{k=0}^{N} r^k$$

- Geometric sum:

$$\sum_{k=0}^{N} r^k = 1 + r + r^2 + \cdots + r^N = \frac{1 - r^{N+1}}{1 - r}$$

- Therefore,

$$\sum_{k=0}^{\infty} r^k = \lim_{N\to\infty} \sum_{k=0}^{N} r^k = \lim_{N\to\infty} \frac{1 - r^{N+1}}{1 - r}$$

Convergence of Geometric Series

- If $|r| > 1$, then geometric series is unbounded
- If $|r| < 1$, then

$$\lim_{n\to\infty} r^n = 0,$$

and therefore

$$\sum_{k=0}^{\infty} r^k = \lim_{N\to\infty} \sum_{k=0}^{N} r^k = \lim_{N\to\infty} \frac{1 - r^{N-n+1}}{1-r} = \frac{1}{1-r}$$